Amenability of von Neumann algebras and spectrum of Dirichlet forms

Fabio Cipriani

Dipartimento di Matematica
Politecnico di Milano

(**joint works with *J.-L. Sauvageot - C.N.R.S. - Paris 7 - Paris 6***)

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Themes:

- Orientation: two classical results about amenability of discrete groups
- Dirichlet forms on Standard Forms of von Neumann algebras
- Caspers-Skalski characterization of Haagerup Approximation Property of von Neumann algebras by spectrum of Dirichlet forms
- Sub-exponential spectral growth of Dirichlet forms and amenability of von Neumann algebras
- Application to amenability of some Compact Quantum Groups
- Relative amenability of inclusions of finite von Neumann algebras
- Conclusions: Dirichlet forms as infinitesimal correspondences

Some references.

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- F.Cipriani, U. Franz, A, Kula, *Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory*, J. Funct. Anal. **266** (2014).
- F. Cipriani, J.-L. Sauvageot *Variations in Noncommutative Potential Theory: finite energy states, potentials and multipliers* Trans. Amer. Math. Soc. **367** (2015).
- R. Okayasu, R. Tomatsu, *Haagerup approximation property for arbitrary von Neumann algebras*, C.R. Math. Acad. Sci. Paris **352** (2014).
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• If Γ is a countable discrete group and ℓ a negative definite type function such that

$$\sum_{g\in\Gamma} e^{-t\ell(g)} < +\infty \qquad \forall\ t > 0$$

then Γ is *amenable* (there exists and invariant state on $l^\infty(\Gamma)$ (J. von Neumann 1929) or there exist positive definite functions $\varphi_n \in c_c(\Gamma)$ pointwise converging to 1.

• Free groups $\mathbb{F}_n$ have the Haagerup Approximation Property (Haagerup 1979) (there exist positive definite functions $\varphi_n \in c_0(\Gamma)$ pointwise converging to 1).

• Γ has the HAP if and only if it admits a *proper* negative definite type functions (Akemann-Walter 1981).

Classical Potential Theory concerns properties of the Dirichlet integral

$$\mathcal{D} : L^2(\mathbb{R}^d, m) \to [0, +\infty] \qquad \mathcal{D}[u] := \int_{\mathbb{R}^d} |\nabla u|^2 \, dm :$$

- l.s.c. quadratic form on $L^2(\mathbb{R}^d, m)$ finite on the Sobolev space $H^1(\mathbb{R}^d)$
- closed form of the Laplace operator $\Delta = -\sum_{k=1}^d \partial_k^2$
- generator of the heat semigroup $e^{-t\Delta}$

The contraction property or Markovianity $\mathcal{D}[u \wedge 1] \leq \mathcal{D}[u]$ is responsible for

- Maximum Principle for harmonic functions $\Delta u = 0$
- positivity and contractivity of $e^{-t\Delta}$ on $L^2(\mathbb{R}^d, m)$, $L^\infty(\mathbb{R}^d, m)$, $L^1(\mathbb{R}^d, m)$.
- Properties above are proved by the knowledge of the Green function

$$\Delta^{-1} u(x) = \int_{\mathbb{R}^d} G(x, y) u(y) \, m(dy) \qquad G(x, y) = |x - y|^{2-d} \qquad d \geq 3 \, .$$

- Beurling and Deny (late '50) developed a kernel free potential theory generalizing the notion of Dirichlet integral to locally compact spaces.
- Fukushima ('60) achieved the construction of the associated Hunt process.

Let $(\mathcal{M}, L^2(\mathcal{M}), L^2_+(\mathcal{M}), J)$ be the standard form of a von Neumann algebra with separable predual and let $\xi_\omega \in L^2_+(\mathcal{M})$ be a fixed cyclic vector representing the f.n. state $\omega \in \mathcal{M}_{*+}$.

Let $\xi \wedge \xi_\omega \in L^2_+(\mathcal{M})$ be the Hilbert projection of a real vector $\xi = J\xi \in L^2(\mathcal{M})$ onto the closed convex set $C_\omega := \xi_\omega - L^2_+(\mathcal{M})$.

Definition. (Dirichlet form)

A Dirichlet form $\mathcal{E} : L^2(\mathcal{M}) \to (-\infty, +\infty]$ is a l.s.c., l.s.b. quadratic form such that

- *the domain $\mathcal{F} := \{\xi \in L^2(\mathcal{M}) : \mathcal{E}[\xi] < +\infty\}$ is dense in $L^2(\mathcal{M})$*
- $\mathcal{E}[J\xi] = \mathcal{E}[\xi]$ *real*
- $\mathcal{E}[\xi \wedge \xi_\omega] \leq \mathcal{E}[\xi]$ *Markovian*
- *$(\mathcal{E}, \mathcal{F})$ is a complete Dirichlet form if its matrix amplification for $n \geq 1$*

$$\mathcal{E}_n[(\xi_{ij})_{ij}] := \sum_{ij} \mathcal{E}[\xi_{ij}]$$

are Dirichlet forms on $\mathcal{M} \otimes \mathbb{M}_n(\mathbb{C})$ (tacitly assumed since now on).

The domain $\mathcal{F}$ is called Dirichlet space when endowed with the graph norm

$$\|\xi\|_{\mathcal{F}} := \sqrt{\mathcal{E}[\xi] + \|\xi\|^2_{L^2(\mathcal{M})}}\,.$$

Definition. (Markovian semigroup)

A self-adjoint C_0-semigroup $\{T_t : t \geq 0\}$ on $L^2(\mathcal{M})$ is ω-Markovian if

- $T_t J = J T_t \qquad t \geq 0$ — *real*
- $\xi \leq \xi_\omega \quad \Rightarrow \quad T_t \xi \leq \xi_\omega \qquad t \geq 0$ — *Markovian*
- $\{T_t : t \geq 0\}$ *on* $L^2(\mathcal{M})$ *is completely Markovian if its matrix amplification*

$$T_t^n([\xi_{ij}]_{ij}) := [T_t(\xi_{ij})]_{ij}$$

are Markovian semigroups on $L^2(\mathcal{M} \otimes \mathbb{M}_n(\mathbb{C}))$ *(tacitly assumed since now on)*

Theorem. (Generalized Beurling-Deny correspondence)

Dirichlet forms are in 1:1 correspondence with Markovian semigroups by

$$\mathcal{E}[\xi] = \lim_{t \to 0} \frac{1}{t}(\xi | \xi - T_t \xi) \qquad T_t = e^{-tL} \qquad \mathcal{E}[\xi] = \|\sqrt{L}\xi\|^2_{L^2(A,\tau)} \qquad a \in \mathcal{F}$$

through the self-adjoint semigroup generator $(L, dom\,(L))$.
In particular, Dirichlet forms are nonnegative $\mathcal{E} \geq 0$ *and Markovian semigroups are positivity preserving and contractive.*

Consider the symmetric embedding $i_\omega : \mathcal{M} \to L^2(\mathcal{M}) \qquad i_\omega(x) := \Delta_\omega^{1/4} x\xi_\omega$
where Δ_ω denotes the modular operator.

Theorem. (Modular ω-symmetry)

Markovian semigroups are in 1:1 correspondence with C_0^*-continuous, positively preserving, contractive semigroups $\{S_t : t \geq 0\}$ on $\mathcal{M}$ which are ω-symmetric

$$\omega(S_t(x)\sigma^\omega_{-i/2}(y)) = \omega(\sigma^\omega_{-i/2}(x)S_t(y)) \qquad x, y \in \mathcal{M}_{\sigma^\omega}\,, \quad t > 0$$

through $\qquad i_\omega(S_t(x)) = T_t(i_\omega(x)) \qquad x \in \mathcal{M}\,.$

- Extending Markovian semigroups from $\mathcal{M}$ to $L^2(\mathcal{M})$ via non symmetric embeddings

  $$i_\alpha(x) := \Delta_\omega^\alpha x\xi_\omega \qquad \alpha \in [0, 1/2] \qquad \alpha \neq 1/4\,,$$

  produces semigroups on $L^2(\mathcal{M})$ which automatically commute with Δ_ω.
- By duality and interpolation, Markovian semigroups extend to C_0-semigroups on noncommutative $L^p(\mathcal{M})$ spaces, $p \in [1, +\infty)$.

Let $\{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous automorphisms group on the C*-algebra A, A_α the algebra of its analytic elements and let $\omega \in A^*_+$ be a KMS$_\beta$-state for $\beta \in \mathbb{R}$.

Definition. (KMS symmetric semigroups on C*-algebras)

A C_0-semigroup $\{S_t : t \geq 0\}$ on A is KMS$_\beta$ symmetric with respect to ω if

$$\omega(bS_t(a)) = \omega(\alpha_{-\frac{i\beta}{2}}(a)S_t(\alpha_{+\frac{i\beta}{2}}(b))) \qquad a, b \in B$$

for some dense, α-invariant, *-subalgebra $B \subseteq A_\alpha$.

- equivalently $\omega(\alpha_{-\frac{i\beta}{2}}(b)S_t(a)) = \omega(\alpha_{-\frac{i\beta}{2}}(a)S_t(b)) \qquad a, b \in B$
- KMS symmetry is a deformation of the KMS condition, in fact for $t = 0$ we get

$$\omega(ba) = \omega(\alpha_{-\frac{i\beta}{2}}(a)\alpha_{+\frac{i\beta}{2}}(b)) = \omega(a\alpha_{+i\beta}(b)) \qquad a, b \in B\,.$$

- In case $\{\alpha_t : t \in \mathbb{R}\}$ and $\{S_t : t \geq 0\}$ commute, KMS symmetry reduces to

$$\omega(bS_t(a)) = \omega(S_t(b)a) \qquad \textit{GNS symmetry}$$

also referred to as detailed balance.

Let $\omega \in A_+^*$ be a KMS$_\beta$-state for $\{\alpha_t : t \in \mathbb{R}\} \subset \mathrm{Aut}\,(A)$ and consider

- the cyclic GNS representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ of A
- the von Neumann algebra $\mathcal{M} := \pi_\omega(A)''$ acting on the space
- $L^2(\mathcal{M}, \omega) \simeq \mathcal{H}_\omega$ carrying
- the standard form determined by $L^2_+(\mathcal{M}, \omega) = \overline{\{\Delta_\omega^{1/4}\pi_\omega(A_+)\xi_\omega\}}$
- the normal extension of ω to $\mathcal{M}$ given by
  $\omega(x) := (\xi_\omega|\pi_\omega(x)\xi_\omega)_2\,, \qquad x \in \mathcal{M}$
- the modular automorphisms group $\{\sigma_t^\omega : t \in \mathbb{R}\}$ of $\mathcal{M}$.

Proposition.

A KMS$_\beta$ symmetric, C_0-semigroup $\{S_t : t \geq 0\}$ on A

- *leaves globally invariant the kernel of the cyclic representation:* $S_t(\ker(\pi_\omega)) \subseteq \ker(\pi_\omega)$
- *extends to a ω-symmetric, C_0^*-semigroup $\{T_t : t \geq 0\}$ on the von Neumann algebra $\mathcal{M}$ by $T_t \circ \pi_\omega = \pi_\omega \circ S_t$*
- *extends to a Markovian semigroup on $L^2(\mathcal{M}, \omega)$*
- *determines a Dirichlet form on the standard form $(\mathcal{M}, L^2(\mathcal{M}, \omega), L^2_+(\mathcal{M}, \omega))$*

- let G be a locally compact, unimodular group with identity $e \in G$
- $\lambda, \rho : G \to \mathcal{B}(L^2(G))$ left, right regular representations
- $C^*_r(G)$ reduced group C*-algebra with trace determined by

$$\tau(a) = a(e) \qquad a \in C_c(G)$$

  and GNS space $L^2(A, \tau) \simeq L^2(G)$
- for any continuous, negative definite function $\ell : G \to [0, +\infty)$

$$\mathcal{E}_\ell[a] = \int_G \ell(g)|a(g)|^2\, dg \qquad a \in L^2(G)$$

  is a Dirichlet form,

$$(T_t a)(t) = e^{-t\ell(g)} a(g) \qquad (La)(g) = \ell(g)a(g) \qquad a \in C_c(G)$$

  are the associated Markovian semigroup and its associated generator.

Let (M,τ) be a nc-probability space and consider

- $1 \in B \subset M$ a *-subalgebra
- $X := \{X_1, \cdots, X_n\} \in M$ nc-random variables, algebraically free w.r.t. B
- $B[X] \subset M$ *-subalgebra generated by X and B
  (regarded as nc-polynomials in the variables X with coefficients in B)
- $W \subset M$ the von Neumann subalgebra generated by $B[X]$.

Theorem. (Voiculescu '00)

There exists unique derivations $\partial_{X_i} : B[X] \to HS(L^2(W,\tau))$ such that

- $\partial_{X_i} X_j = \delta_{ij} 1 \otimes 1\,, \qquad i,j = 1, \cdots, n;$
- $\partial_{X_i} b = 0 \qquad i = 1, \cdots, n, \quad b \in B\,.$

Under the assumption $1 \otimes 1 \in \mathrm{dom}\,(\partial^*_{X_i})$ for all $i = 1, \cdots, n$, it follows that

- *$(\partial_{X_i}, B[X])$ is closable in $L^2(W,\tau)$ for all $i = 1, \cdots, n$,*
- *the closure of $\mathcal{E}_X[a] := \sum_{i=1}^n \|\partial_{X_i} a\|^2$ is a Dirichlet form.*

A second countable, locally compact group G has the Haagerup Approximation property HAP if there exists a sequence of normalized, positive definite functions, vanishing at infinity $\varphi_n \in C_0(G)$, converging to the constant function 1, uniformly on compact subsets.

Equivalently, G has the HAP if there exists a proper, continuous, negative definite function on G.

By a result of U. Haagerup ('79), the free groups $\mathbb{F}_n$ have the HAP as their length functions are negative definite.

A long research (Connes-Jones, Choda, Jolissaint, Boca, Popa) culminated with various definitions of the HAP valid in general von Neumann algebras.

Let us consider the following one.

Definition. (Okayasu-Tomatsu 2014)

A von Neumann algebra $\mathcal{M}$ has the HAP if there exists a standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, \mathcal{J})$ and a sequence of contractive, completely positive, compact operators $T_n : \mathcal{H} \to \mathcal{H}$ such that $\|\xi - T_n\xi\|_{\mathcal{H}} \to 0$ as $n \to +\infty$, for all $\xi \in \mathcal{H}$.

Recently the HAP has been characterized in terms of Dirichlet forms.

Theorem. (Caspers-Skalski 2015)

The following properties are equivalent

- *The von Neumann algebra $\mathcal{M}$ with separable predual has the HAP*
- *there exists a Markovian semigroup $\{T_t : t \geq 0\}$ (w.r.t. a cyclic vector $\xi_\omega \in \mathcal{P}$) such that T_t is compact for all $t > 0$*
- *there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ (w.r.t. a cyclic vector $\xi_\omega \in \mathcal{P}$) having discrete spectrum .*

As an application one can provide an alternative proof of the following result.

Corollary. (Brannan 2012)

The von Neumann algebras $L^\infty(C_r(O_N^+), h)$ of the free orthogonal quantum groups O_N^+ in the cyclic representation of the Haar state h on $L^2(C_r(O_N^+), h)$, have the HAP.

Proof. The result follows from the Caspers-Skalski equivalence and the construction of a suitable Dirichlet form with discrete spectrum done in [CFK 2014].

Definition. (Amenable von Neumann algebras)

A von Neumann algebra N is amenable if all normal derivations of N with coefficients in any dual Banach N-bimodule X are inner *in the sense that*

$$\delta(x) = x\eta - \eta x \qquad x \in N$$

from some fixed vector $\eta \in X$.

Amenability is equivalent to other fundamental properties:
injectivity, semi-discreteness, property E, property P, hyperfiniteness.

A useful point of view on amenability relies on the following generalized notion of morphism between von Neumann algebras, introduced by A. Connes.

Definition. (Correspondences)

Let M and N be von Neumann algebras. A correspondence from M to N is a Hilbert space $\mathcal{H}$ which is an $N - M$-bimodule.

In other words, a correspondence from M to N can be seen both

- as an Hilbert space $\mathcal{H}$ supporting two commuting, normal, $*$-representations π_N of N and π_{M° of (the opposite algebra) M° and
- as a representation π of the C*-algebra $N \otimes_{\max} M^\circ$ which is *binormal* in the sense that its restrictions to N and M° are normal representations.

The latter point of view allows to translate to correspondences, notions from representations theory, notably *containment* and *weak containment*

Identity or standard correspondence. On the Hilbert space $L^2(N)$ of the standard form of N, this correspondence is given by the representation of $N \otimes_{\max} N^\circ$ defined

$$\pi_{\mathrm{id}}(x \otimes y^\circ)\xi := x\xi y := xJ_n y^* J_N \xi \qquad x, y \in N,\ \xi \in L^2(N)\,.$$

Coarse or Hilbert-Schmidt correspondence. On the tensor product of Hilbert spaces $L^2(N) \otimes L^2(N)$, this correspondence is given by

$$\pi_{\mathrm{co}}(x \otimes y^\circ)(\xi \otimes \eta) := x\xi \otimes \eta y = x(\xi \otimes \eta)y \qquad x, y \in N,\ \xi, \eta \in L^2(N)\,.$$

Correspondence associated to a completely positive map. Let $T : L^2(N) \to L^2(N)$ be a completely positive map such that $T\xi_\omega = \xi_\omega$ and consider the state

$$\Phi_T : N \otimes_{\max} N^\circ \to \mathbb{C} \qquad \Phi_T(x \otimes y^\circ) := (i_\omega(y^*) | T\, i_\omega(x))_{L^2(N)} \qquad x, y \in N\,.$$

The associated correspondence $\mathcal{H}_T$ is given by the GNS construction applied to Φ_T.

- Correspondences from N to N behave in a way similar to *representations of groups*, once *tensor product* of representations is replaced by the *composition* or *relative tensor product* of correspondences:
- the identity correspondence $\mathcal{H}_{\text{id}}$ plays the role of the trivial representation
- the coarse correspondence $\mathcal{H}_{\text{co}}$ plays the role of the left regular representation

This suggested to state and investigate Property T for von Neumann algebras (Connes-Jones '85) and

Definition. *Amenability by correspondences* (Popa '86)

A von Neumann algebra N is amenable in the sense of correspondences if the identity correspondence $\mathcal{H}_{\text{id}}$ is weakly contained in the coarse correspondence $\mathcal{H}_{\text{co}}$.

It has been observed that amenability in the above sense is equivalent to injectivity (Popa 86', Anantharaman-Delaroche '90).

Let $(L, D(L))$ be the selfadjoint operator of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \omega)$.

Definition. (Spectral growth rate)

Whenever the spectrum of $(L, D(L))$ is discrete $\sigma(L) := \{\lambda_k \geq 0 : k \in \mathbb{N}\}$, set

$$\Lambda_n := \{k \in \mathbb{N} : \lambda_k \in [n, n+1)\}, \qquad m_n := \sharp(\Lambda_n) \qquad n \in \mathbb{N}$$

and define the spectral growth rate as $\omega(\mathcal{E}, \mathcal{F}) := \limsup_{n\in\mathbb{N}} \sqrt[n]{m_n}$.
The Dirichlet space $(\mathcal{E}, \mathcal{F})$ is said to have
- *exponential spectral growth if $\omega(\mathcal{E}, \mathcal{F}) > 1$*
- *sub-exponential spectral growth if $\omega(\mathcal{E}, \mathcal{F}) \leq 1$.*

Example. In case of Dirichlet forms $(\mathcal{E}_\ell, \mathcal{F}_\ell)$ associated to negative definite length functions ℓ on finitely generated countable discrete groups Γ, the spectral growth rate $\omega(\mathcal{E}_\ell, \mathcal{F}_\ell)$ reduces to the growth rate $\omega(\Gamma)$.

Lemma. (Nuclear Markovian semigroups)

The Dirichlet space $(\mathcal{E}, \mathcal{F})$ has sub-exponential growth rate if and only if the Markovian semigroup $\{e^{-tL} : t > 0\}$ on $L^2(N, \omega)$ is nuclear in the sense that

$$\text{Trace}\,(e^{-tL}) = \sum_{k\in\mathbb{N}} e^{-t\lambda_k} < +\infty \qquad \forall\ t > 0\,.$$

Theorem. (Sub-exponential spectral growth rate and amenability)

Let N be a von Neumann algebra. If there exists a faithful normal state $\omega \in N_{*+}$ and a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(N,\omega)$ having sub-exponential spectral growth rate, then N is amenable in the sense of correspondences.

Sketch of Proof. Assume for simplicity that $\mathcal{E}[\xi_\omega]=0$ so that $T_t\xi_\omega=\xi_\omega$ for $t>0$. Consider the cyclic correspondences $\mathcal{H}_t$ associated to the completely positive maps T_t determined by the binormal states

$$\Phi_t : N\otimes_{\max} N^\circ \to \mathbb{C} \qquad \Phi_t(x\otimes y^\circ) := (i_\omega(y^*)|T_t\, i_\omega(x))_{L^2(N)}\,.$$

As T_t commutes with J_N, assume the eigenvectors ξ_k associated to the eigenvalues λ_k to be real: $\xi_k=J_N\xi_k$. Since $T_t=\sum_{k\in\mathbb{N}} e^{-t\lambda_k}\xi_k\otimes\overline{\xi}_k$ we have

$$\Phi_t(x\otimes y^\circ) = (i_\omega(y^*)|T_t i_\omega(x)) = \sum_{k=0}^{\infty} e^{-t\lambda_k}(i_\omega(y^*)|(\xi_k\otimes\overline{\xi_k})i_\omega(x))$$

$$= \sum_{k=0}^{\infty} e^{-t\lambda_k}(i_\omega(y^*)|(\xi_k|i_\omega(x))\xi_k) = \sum_{k=0}^{\infty} e^{-t\lambda_k}(\xi_k|i_\omega(x))(i_\omega(y^*)|\xi_k)$$

$$= \sum_{k=0}^{\infty} e^{-t\lambda_k}(\xi_k|i_\omega(x))(J_N\xi_k|J_N i_\omega(y^*)) = \sum_{k=0}^{\infty} e^{-t\lambda_k}(\xi_k|i_\omega(x))(J_N\xi_k|i_{\omega^\circ}(y^\circ))$$

$$= \sum_{k=0}^{\infty} e^{-t\lambda_k} (\xi_k \otimes J_N \xi_k | i_\omega(x) \otimes i_{\omega^\circ}(y^\circ))_{L^2(N,\omega)\otimes L^2(N,\omega)}$$

$$= \left(\sum_{k=0}^{\infty} e^{-t\lambda_k} \xi_k \otimes \xi_k \middle| i_\omega(x) \otimes i_{\omega^\circ}(y^\circ) \right)_{L^2(N,\omega)\otimes L^2(N,\omega)}$$

$$= \left(Z_t \middle| i_{\omega\otimes\omega^\circ}(x \otimes y^\circ) \right)_{L^2(N\overline{\otimes}N^\circ, \omega\otimes\omega^\circ)}$$

where the series $Z_t := \sum_{k=0}^{\infty} e^{-t\lambda_k} \xi_k \otimes \xi_k \in L^2(N,\omega) \otimes L^2(N,\omega)$ converges for all $t > 0$ by the nuclearity of the semigroup. Since the symmetric embedding $i_{\omega\otimes\omega^\circ}$ is weakly*-continuous, the identity above proves that Φ_t is a normal state on $N\overline{\otimes}N^\circ$. By the properties of the standard forms, $\exists\ \Omega_t \in L^2(N\overline{\otimes}N^\circ, \omega \otimes \omega^\circ)_+$ such that

$$\Phi_t(z) = (i_\omega(y^*) | T_t i_\omega(x)) = \left(\Omega_t | \pi_{\mathrm{co}}(z) \Omega_t \right)_{L^2(N\overline{\otimes}N^\circ, \omega\otimes\omega^\circ)} \qquad z \in N \otimes_{\max} N^\circ$$

so that the correspondence $\mathcal{H}_t$ is contained in the coarse correspondence $\mathcal{H}_{\mathrm{co}}$. Since the semigroup is strongly continuous on $L^2(N,\omega)$

$$\lim_{t\downarrow 0} \left(\Omega_t | \pi_{\mathrm{co}}(z) \Omega_t \right)_{L^2(N,\omega)\otimes L^2(N,\omega)} = (\xi_\omega | \pi_{\mathrm{id}}(z) \xi_\omega) \qquad z \in N \otimes_{\max} N^\circ$$

so that the identity correspondence $\mathcal{H}_{\mathrm{id}}$ is weakly contained in the coarse correspondence $\mathcal{H}_{\mathrm{co}}$ and N is amenable in the sense of correspondences.

- The C*-algebra $C_u(O_N^+)$ is generated by $\{v_{jk} = v_{jk}^* : i,k = 1,\cdots,N\}$ subject to

$$\sum_{l=1}^{N} v_{lj} v_{lk} = \delta_{jk} = \sum_{l=1}^{N} v_{jl} v_{kl} \qquad \Delta v_{jk} = \sum_{l=1}^{N} v_{lj} \otimes v_{lk}$$

- classes of irreducible, unitary corepresentations $\widehat{O_N^+} \cong \mathbb{N}$
- the Haar h state is a trace, faithful on $\mathrm{Pol}(O_N^+)$ but not on $C_u(O_N^+)$

In [CFK '14], a Lévy process on the reduced C*-algebra $C_r(O_N^+)$ has been constructed by a

- *translation invariant* Markovian semigroup e^{-tL}

$$\Delta \circ T_t = (\mathrm{id} \otimes T_t) \circ \Delta \qquad t > 0$$

with generating functional $G = \varepsilon \circ L$.

As an application one can provide an alternative proof of the following result.

Corollary 1. (Brannan 2012)

The von Neumann algebras $L^\infty(C_r(O_2^+), h)$ of the free orthogonal quantum group O_2^+ in the cyclic representation of the Haar state h on $L^2(C_r(O_2^+), h)$ is amenable.

Proof.

- denote $U_s \in \mathrm{Pol}[-N, N]$ the Chebyshev polynomial of the second kind

$$U_0(x) = 1, \quad U_1(x) = x, \quad U_s(x) = xU_{s-1}(x) - U_{s-1}(x), \qquad x \in [-N, N], \quad s \in \mathbb{N}$$

- generating functional $G(u_{jk}^{(s)}) := \delta_{jk} \frac{U_s'(N)}{U_s(N)}, \quad s \in \mathbb{N}, \quad j, k = 1, \cdots, U_s(N)$
- the generator has discrete spectrum, eigenvalues and multiplicities are given by

$$\lambda_s = \frac{U_s'(N)}{U_s(N)}, \qquad m_s = (U_s(N))^2$$

Let $B \subseteq N$ be an inclusion of finite von Neumann algebras with trace τ.

Let $E_B : N \to N$ be the conditional expectation onto B and $e_B \in \text{Proj}\,(L^2(N,\tau))$ the positive preserving projection onto the subspace $L^2(B,\tau)$

$$e_B(x\xi_\tau) := E_B(x)\xi_\tau \qquad x \in N\,.$$

Denote by $\mathcal{H}_B$ the correspondence associated to e_B, determined by the binormal state

$$\Phi_B : N \otimes_{\max} N^\circ \to \mathbb{C} \qquad \Phi_B(x \otimes y^\circ) := \tau(E_B(x)y)\,.$$

Definition. *Popa '86* (Relative amenability)

The inclusion $B \subseteq N$ is amenable if $\mathcal{H}_{\mathrm{id}}$ is weakly contained in $\mathcal{H}_B$.

To state a condition in terms of a spectral properties of Dirichlet forms on $L^2(N,\tau)$ providing relative amenability, let us recall the basic construction (Christensen, Skau, Jones, Pimsner-Popa).

Let us denote by $\langle N, B\rangle$ the von Neumann algebra in $\mathcal{B}(L^2(N,\tau))$ generated by N and the projection e_B.

Examples. If $B = \mathbb{C}1_N$ then $\langle N, B\rangle = \mathcal{B}(L^2(N,\tau))$ and if $B = N$ then $\langle N, B\rangle = N$.

Denoting by $\xi_\tau \in L^2(N,\tau)$ the cyclic vector representing τ one has

$$e_B(x\xi_\tau) = E_B(x)\xi_\tau\,, \quad e_B x e_B = E_B(x)e_B \qquad x \in N\,.$$

An element $x \in N$ commutes with the projection, $xe_B = e_B x$, if and only if $x \in B$.

Moreover, $\mathrm{span}(Ne_BN)$ is weakly*-dense in $\langle N,B\rangle$ and $e_B\langle N,B\rangle e_B = Be_B$.

It can be shown that

$$\langle N,B\rangle = (J_N B J_N)' = J_N B' J_N \subseteq \mathcal{B}(L^2(N,\tau))\,.$$

There exists a unique normal, semifinite faithful trace Tr on $\langle N,B\rangle$ characterized by

$$\mathrm{Tr}(xe_By) = \tau(xy) \qquad x,y \in N\,.$$

There exists also a unique N-bimodule map Φ from $\mathrm{span}(Ne_BN) \to N$ satisfying

$$\mathrm{Tr} = \tau\circ\Phi \qquad \Phi(xe_By) = xy \quad x,y \in N\,.$$

This map extends to a contraction of $N-N$-bimodules

$$\Phi : L^1(\langle N,B\rangle,\mathrm{Tr}) \to L^1(N,\tau) \qquad \text{and} \qquad e_BX = e_B\Phi(e_BX) \qquad X \in \langle N,B\rangle\,.$$

Moreover, $\Phi(e_BX) \in L^2(\langle N,B\rangle,\mathrm{Tr})$ for all $X \in \langle N,B\rangle$.

These properties enable us to prove that the identity correspondence $L^2(\langle N, B\rangle, \mathrm{Tr})$ of $\langle N, B\rangle$ reduces to the relative correspondence $\mathcal{H}_B$ when restricted to the subalgebra $N \subseteq \langle N, B\rangle$.

Proposition.

The $N - N$-correspondences $\mathcal{H}_B$ and $L^2(\langle N, B\rangle, \mathrm{Tr})$ are isomorphic. In particular, the binormal state is given by

$$\Phi_B(x \otimes y^\circ) = (e_B|xe_By)_{L^2(\langle N,B\rangle,\mathrm{Tr})} \qquad x, y \in N$$

so that the cyclic vector representing the state Φ_B is $e_B \in L^2(\langle N, B\rangle, \mathrm{Tr})$.

Definition. (*B*-invariant Dirichlet forms)

A Dirichlet form $(\mathcal{E},\mathcal{F})$ *on* $L^2(N,\tau)$ *is B-invariant if*

$$b(\mathcal{F})\subseteq\mathcal{F}\,,\quad \mathcal{E}[b(\xi)]=\mathcal{E}(b^*b\xi|\xi)=\mathcal{E}(\xi|\,\xi bb^*)\qquad b\in B,,\quad \xi\in\mathcal{F}\,.$$

Alternatively, B-invariance means that the semigroup is for some and hence all $t>0$ *a B-bimodular map*

$$e^{-tL}\circ L(b)=L(b)\circ e^{-tL}$$
$$e^{-tL}\circ R(b)=R(b)\circ e^{-tL}\qquad b\in B$$

(where L,R *are the left and right standard actions of* N *on* $L^2(N,\tau)$*). By Markovianity* $e^{-tL}\circ J_N=J_N\circ e^{-tL}$ *for all* $t>0$ *so that B-invariance implies*

$$e^{-tL}\in(J_NBJ_N)'\cap B'=\langle N,B\rangle\cap B'\qquad t>0\,.$$

Example 1. Any Dirichlet form is *ker* $(\mathcal{E})$-invariant: $P_{ker\,(\mathcal{E})}=\lim_{t\to0}e^{-tL}$.

Example 2. Voiculescu's Dirichlet form is B-invariant.

Example 3. The quadratic form $\mathcal{E}_D$ of the Dirac Laplacian D^2 of compact Riemannian manifolds is *ker* (D)-invariant with respect *harmonic forms*, provided the curvature operator is nonnegative $\hat{R}\geq0$.

Recall that the *compact ideal space* $\mathcal{J}(\langle N, B\rangle)$ is the norm closed, two sided ideal generated by finite projections of the semifinite von Neumann algebra $\langle N, B\rangle$.

Examples. $\mathcal{J}(\langle N, \mathbb{C}1_N\rangle) = \mathcal{K}(L^2(N,\tau))$ and $\mathcal{J}(\langle N, N\rangle) = N$.

Definition. (Relative discrete spectrum and its growth)

$(\mathcal{E},\mathcal{F})$ or $(L, D(L))$ have *discrete spectrum relative to* $B \subseteq N$ *if*

$$e^{-tL} \in \mathcal{J}(\langle N, B\rangle) \qquad \text{for some and hence all } t > 0\,.$$

$(\mathcal{E},\mathcal{F})$ *is said to have*

- *exponential growth relative to* $B \subseteq N$ *if* $\mathrm{Tr}(e^{-tL}) = +\infty$ for some $t > 0$;
- *subexponential growth relative to* $B \subseteq N$ *if* $\mathrm{Tr}(e^{-tL}) < +\infty$ *for all* $t > 0$.

Theorem. (Relative sub-exponential spectral growth rate and relative amenability)

Let $B \subseteq N$ be an inclusion of finite von Neumann algebras with trace τ. If there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(N, \tau)$ having sub-exponential spectral growth relatively to $B \subseteq N$, then the inclusion $B \subseteq N$ is amenable.

The proof is based on the above properties of the basic construction and on the following formula

$$(T^*|xe_By)_{L^2(\mathrm{Tr})} = (i_\tau(y^*)|T(i_\tau(x))_{L^2(\tau))} \qquad T \in \langle N, b\rangle \cap L^2(\langle N, b\rangle, \mathrm{Tr}), \quad x, y \in N.$$

As $\mathrm{span}(Ne_BN)$ is weakly* dense in $\langle N, b\rangle$, it is enough to prove the formula for $T \in Ne_BN$. If $T = ue_Bv$ for some $u, v \in N$ one uses the identity

$$e_ByTxe_B = e_Byue_Bvxe_B = (e_Byue_B)(e_Bvxe_B) = E_B(yu)e_BE_B(vx)e_B.$$

Then the formula can be applied with $T = e^{-tL}$ for all $t > 0$ since the hypotheses of relative sub-exponential spectral growth rate $e^{-tL} \in L^2(\langle N, B\rangle, \mathrm{Tr})$.