INVARIANTS FROM KK-THEORY

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Overview

Lecture 1. Summary of KK theory.

Recommended reading: “A primer on KK-theory" Nigel Higson (appeared in Proceedings of Symposia in Pure Mathematics, Vol 51 (1990) Part I.)

Downloadable from Nigel’s website.

Much better than this lecture.

Lecture 2. (a) Models in condensed matter theory.

(b) Application to bulk-edge correspondence in the integer quantum Hall effect. arXiv:1411.7527

Lecture 3. Topological phases and KKO. 1509.07210

Kasparov's bivariant theory

I want to start with the definition in the complex case which will need some unpacking. Note that I will work with ungraded algebras for simplicity.

Definition. Let A and B be C^*-algebras, with A separable. An odd Kasparov A-B-module X consists of a countably generated ungraded right B-C^*-module X, with $\pi : A \to \mathrm{End}_B(X)$ a $*$-homomorphism, together with $F \in \mathrm{End}_B(X)$ such that

$$\pi(a)(F - F^*),\ \pi(a)(F^2 - 1),\ [F, \pi(a)]$$

are compact adjointable endomorphisms of X, for each $a \in A$.

Additional definitions, notation...

X is a left A, right B module.

As a B-module it is a C^*-module (also called a 'Hilbert C^*-module) meaning it is equipped with a right-action and a B valued inner product: $X \times X \to B$, written as $x, y \mapsto (x|y)_B \in B$. It is conjugate linear in the first variable linear in the second.

The inner product satisfies some additional axioms that generalise the notion of an inner product so that the complex numbers are replaced in a sense by the algebra (noncommutative in general) B.

These are tedious to write down. Most important are

$$(xb|y)_B = b^*(x|y)_B \text{ and } (x|yb)_B = (x|y)_B b$$

For example in the case where $X = B$ then $(x|y)_B = x^*y$.

Rank one operators:

$$\theta_{x,y} z = x \cdot (y|z)_B \quad x, y, z \in X$$

Finite rank operators are finite linear combinations of rank one operators.

Norm on X is given by $\|x\|^2 = \|(x|x)_B\|$.

X must be both complete and countably generated as a B module. Thus B is a Banach space.

This results in there being important technical differences from Hilbert space theory later on when we study Kasparov modules.

$\mathrm{End}_B(X)$ consists of the B-linear endomorphisms of X.

Compact endomorphisms: close up the finite rank operators in the norm on continuous linear operators on X.

An adjointable operator T on X is one where there is an operator $T^* : X \to X$ with

$$(T^* x|y)_B = (x|Ty)_B.$$

Not all bounded operators on X are adjointable.

An even Kasparov A-B-module has all the previous structure together with a $\mathbf{Z}_2$ grading operator. This means we have a self-adjoint B-linear endomorphism γ with $\gamma^2 = 1$ and $\pi(a)\gamma = \gamma\pi(a)$, $F\gamma + \gamma F = 0$.

We will use the notation $({}_AX_B, F)$ or $({}_AX_B, F, \gamma)$ for Kasparov modules, generally omitting the representation π. A Kasparov module $({}_AX_B, F)$ with $\pi(a)(F - F^*) = \pi(a)(F^2 - 1) = [F, \pi(a)] = 0$, for all $a \in A$, is called *degenerate*.

The KK-group

We now describe the equivalence relation on Kasparov A-B-modules which defines classes in the abelian group $KK(A,B) = KK^0(A,B)$ (even case) or $KK^1(A,B)$ (odd case). Because of Bott periodicity there are only these two groups.

The relation consists of three separate equivalence relations: unitary equivalence, stable equivalence and operator homotopy.

Two Kasparov A-B-modules $({}_A(X_1)_B, F_1)$ and $({}_A(X_2)_B, F_2)$ are *unitarily equivalent* if there is an adjointable unitary B-module map $U : X_1 \to X_2$ such that $\pi_2(a) = U\pi_1(a)U^*$, for all $a \in A$ and $F_2 = U\,F_1\,U^*$.

Two Kasparov A-B-modules $({}_A(X_1)_B, F_1)$ and $({}_A(X_2)_B, F_2)$ are *stably equivalent* if there is a degenerate Kasparov A-B-module $({}_A(X_3)_B, F_3)$ with $({}_A(X_1)_B, F_1) = ({}_A(X_2 \oplus X_3)_B, F_2 \oplus F_3)$ and $\pi_1 = \pi_2 \oplus \pi_3$.

Thus the degenerate modules play the role that is taken by trivial vector bundles in the Atiyah-Hirzebruch K-theory.

Two Kasparov A-B-modules $({}_A(X)_B, G)$ and $({}_A(X)_B, H)$ (with the same representation π of A) are called *operator homotopic* if there is a norm continuous family $(F_t)_{t\in[0,1]} \subset \mathrm{End}_B(X)$ such that for each $t \in [0, 1]$ $({}_A(X_1)_B, F_t)$ is a Kasparov module and $F_0 = G$, $F_1 = H$.

Two Kasparov modules $({}_A(X)_B, G)$ and $({}_A(X')_B, G')$ are equivalent if after the addition of degenerate modules, they are operator homotopic to unitarily equivalent Kasparov modules.

The equivalence classes of even (resp. odd) Kasparov A-B modules form an abelian group denoted $KK^0(A, B)$ (resp. $KK^1(A, B)$).

The zero element is represented by the degenerate Kasparov modules.

The inverse of a class $[({}_A(X)_B, F)]$ is the class of $({}_A(X)_B, -F)$, with grading $-\gamma$ in the even case.

Special cases:

$KK^*(\mathbf{C}, C)$ is the K-group of C.

$KK^*(A, \mathbf{C})$ is the K-homology group of A due essentially to Kasparov but is based on Atiyah's Ell-theory.

As the algebra B is replaced by the complex numbers here representatives of this group are Hilbert spaces that are A modules.

Kasparov product

The deepest part of the theory developed by Kasparov is the product

$$KK^i(A,B) \times KK^j(B,C) \mapsto KK^{i+j}(A,C).$$

The Kasparov product leads to the notion of $KK^0(A,B)$ as a 'morphism' between A and B because we have:

(i) for example,

$$KK^0(\mathbf{C},A) \times KK^0(A,B) \mapsto KK^0(\mathbf{C},B).$$

That is we may think of $KK^0(A,B)$ as mapping the K-theory of A to the K-theory of B.

(ii) and the 'composition rule'

$$KK^0(A,B) \times KK^0(B,C) \mapsto KK^0(A,C).$$

Meyer-Nest formalised this by introducing the triangulated category whose objects are C^*-algebras and whose morphisms from A to B are given by the elements of $KK^0(A, B)$.

The duality of K-theory and K-homology is also captured by the Kasparov product:

$$KK^0(\mathbf{C}, B) \times KK^0(B, \mathbf{C}) \mapsto KK^0(\mathbf{C}, \mathbf{C}) \cong \mathbf{Z}.$$

The Kasparov product and the index pairing in this form are what are used in the bulk-edge correspondence argument that is coming up.

2. Fredholm modules and K-homology.

We specialise to the case of $KK(\mathcal{A}, \mathbf{C})$.

Definition. Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. A *Fredholm module over* $\mathcal{A}$ is given by a Hilbert space $\mathcal{H}$, a $*$-representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a bounded operator $F : \mathcal{H} \to \mathcal{H}$ such that for all $a \in \mathcal{A}$:

$$(F^2 - 1)\rho(a), \;\; (F - F^*)\rho(a), \;\; [F, \rho(a)] := F\rho(a) - \rho(a)F$$

are all compact operators.

We say that $(\rho, \mathcal{H}, F)$ is even (or graded) if there is an operator $\gamma : \mathcal{H} \to \mathcal{H}$ such that $\gamma^2 = 1$, $\gamma = \gamma^*$, $\gamma F + F\gamma = 0$ and for all $a \in \mathcal{A}$, $\gamma\rho(a) = \rho(a)\gamma$. Otherwise we call $(\rho, \mathcal{H}, F)$ odd.

Spectral triples, the unbounded picture

Sometimes called K-cycles these provide an alternative construction of K-homology that is not equivalent to the Fredholm module picture.

The basic data consists of:

(i) $\mathcal{H}$, a separable Hilbert space
(ii) $\mathcal{D}$ is a densely defined self-adjoint operator on $\mathcal{H}$
(iii) $\mathcal{A}$ is a *-subalgebra of the algebra of bounded operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$.

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple if $[\mathcal{D}, a]$ is bounded and $a(1+\mathcal{D}^2)^{-1/2}$ is a compact operator for all $a \in \mathcal{A}$.

If there is a grading $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma\mathcal{D} + \mathcal{D}\gamma = 0$ then we say the spectral triple is even and otherwise it is odd.

Spectral triples are unbounded K-homology classes.

Semifinite spectral triples

In fact there is a more general notion which is proving useful in some applications.

This is where we replace $\mathcal{B}(\mathcal{H})$ by a semifinite von Neumann subalgebra $\mathcal{N}$ of $\mathcal{B}(\mathcal{H})$ with $\mathcal{A} \subset \mathcal{N}$.

Of course we also need to replace the compact operators on $\mathcal{H}$ by the compact operators $\mathcal{K}(\mathcal{N})$ in $\mathcal{N}$ and then the previous definition can be carried over.

These semifinite spectral triples also fit into KK-theory as in the Kaad-Nest-Rennie paper which shows they define elements of $KK(\mathcal{A}, \mathcal{K}(\mathcal{N}))$.

More precisely they prove that spectral flow for self adjoint Fredholm operators in a semifinite von Neumann algebra comes from elements in $KK^1(\mathcal{A}, \mathcal{K}(\mathcal{N}))$.

Connection to Fredholm modules

Proposition

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be a spectral triple. Define (Riesz map)*

$$F_{\mathcal{D}} = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}.$$

Then $(\mathcal{H}, F_{\mathcal{D}})$ *is a Fredholm module for the* C^**-algebra* $A := \overline{\mathcal{A}}$*.*

Further KK properties

$KK(*,*)$ is a bifunctor that takes pairs of algebras to abelian groups. It is covariant in the second variable and contravariant in the first variable.

$KK(*,*)$ is 'stable' in the sense that if we tensor A or B by a copy of the compact operators on a separable Hilbert space then we do not change $KK^*(A,B)$.

There are a number of ways to express *Bott periodicity* in KK-theory. Kasparov exploited Clifford algebra periodicity (period 2 in the complex case and period 8 in the real case) to achieve this.

Thus in the complex case we consider $KK(A \otimes C\ell_n, B \otimes C\ell_m)$ where $C\ell_n$ is the complex Clifford algebra on n generators. (I am assuming ungraded algebras A, B.) But there are only two non-isomorphic groups due to the properties of Clifford algebras.

The real case

If A, B are real algebras then we can define real Kasparov A, B bimodules by modifying the complex definition in the obvious way.

Let $C\ell_{p.q}$ be the real Clifford algebra (the signature of the quadratic form defining the algebra is p, $+$'s and q, $-$'s) then we have the higher KKO groups:

$$KKO(A \otimes C\ell_{m.n}, B \otimes C\ell_{p,q}).$$

One obtains only 8 non-isomorphic KKO groups as there are only 8 Clifford algebras, up to Morita equivalence, in the real case.

Also we can 'move the Clifford algebra', for example

$$KKO(A \otimes C\ell_{n,0}, B) \cong KKO(A, B \otimes C\ell_{0,n}).$$

KKO seems to be essential for index theory for the 'topological phases of matter' question.

KK-equivalence

Two C^*-algebras A and B are KK-equivalent if there exist elements $\alpha \in KK(A, B)$ and $\beta \in KK(B, A)$ such that the Kasparov product $\alpha \times \beta$ is the identity in $KK(A, A)$ and $\beta \times \alpha$ is the identity in $KK(B, B)$.

KK-equivalence implies that the K-groups of A and B are the same in a functorial way.

Application: by Kirchberg's work, simple, purely infinite algebras that are KK-equivalent and in the bootstrap class are isomorphic.

Further equivalences from Kasparov modules

Self-Morita equivalence bimodules:

An Hilbert C^*-module is full if the range of the inner product is dense.

Let A be a unital C^*-algebra. A bi-Hilbertian A-bimodule is a full right C^*-A module with inner product $(\cdot|\cdot)_A$ which is also a full left Hilbert A-module with inner product ${}_A(\cdot|\cdot)$ such that the left action of A is adjointable with respect to $(\cdot|\cdot)_A$ and the right action of A is adjointable with respect to ${}_A(\cdot|\cdot)$

A self-Morita equivalence bimodule over A is a bi-Hilbertian A-bimodule E whose inner products are both full and satisfy the imprimitivity condition ${}_A(e|f)g = e(f|g)_A$, for all $e, f, g \in E$.

The equivariant theory

We assume G is a compact group and A, B have continuous G-actions.

Definition

The group $KK_G^*(A, B)$ is defined by taking equivalence classes of Kasparov A, B bimodules X such that
(i) the module X is equipped with an action of G such that $(g.a_1, g.a_2)_B = g.[(a_1, a_2)_B]$ for all $a_1, a_2 \in X, g \in G$,
(ii) the actions of A and B on X are G-equivariant,
(iii) the operator F commutes with the G-action.

When G is not compact the definition is more complicated.

Finite symmetry groups are relevant to the applications to condensed matter models and these are needed to give a KK approach to the more complicated parts of the Freed-Moore analysis of topological phases (this has not been written out).

$\mathbb{T}$-equivariance has also played a role in much recent work (cf C-Neshveyev-Nest-Rennie).

The unbounded version

Tricky point: the functional calculus in Hilbert C^*-modules.

Because our Hilbert C^*-modules are only Banach spaces the functional calculus is not always available.

The algebra of operators for which it works are termed 'regular' (note over use of the term regular).

Reference: The functional calculus of regular operators on Hilbert C^*-modules revisited J. Kustermans arXiv 9706007.

Definition

If X is a B-C^*-module and T is a densely defined B-linear mapping from X to X such that T is closed, T^* is densely defined and $1 + T^*T$ has dense range then T is called regular.

Definition

Given $\mathbf{Z}_2$-graded C^*-algebras A and B, an even *unbounded* Kasparov A-B-module $({}_A(X)_B, \mathcal{D})$ is given by

1. A $\mathbf{Z}_2$-graded, countably generated, right B C^*-module X_B;
2. A $\mathbf{Z}_2$-graded $*$-homomorphism $\rho\colon A \to \mathrm{End}_B(X)$;
3. A self-adjoint, regular, odd operator $\mathcal{D}\colon \mathrm{Dom}(\mathcal{D}) \subset X \to X$ such that $[\mathcal{D}, \rho(a)]_\pm$ is an adjointable endomorphism, and $\rho(a)(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism for all a in a dense subalgebra $\mathcal{A}$ of A.

If the module and algebras are trivially graded, then the Kasparov module is called odd.

Proposition (Baaj-Julg)

If $({}_A(X)_B, \mathcal{D})$ is an unbounded Kasparov module, then $\left({}_A(X)_B, \mathcal{D}(1+\mathcal{D}^2)^{-1/2}\right)$ is a Kasparov module.

Recent work by Bram Mesland, Jens Kaad, Matthias Lesch, ... has shown that in many examples the product can be written explictly using unbounded Kasparov modules. That is we choose unbounded representatives of classes in each of the first two groups and combine them to produce the third representative:

$$KK(A,B) \times KK(B,\mathbf{C}) \to KK(A,\mathbf{C})$$
$$[({}_A(X)_B,\mathcal{D}_1)]\,\hat{\otimes}_B\,[({}_B(\mathcal{H})_{\mathbf{C}},\mathcal{D}_2)] = \left[\left({}_A(X\hat{\otimes}_B\mathcal{H})_{\mathbf{C}},\mathcal{D}_1\hat{\otimes}1 + 1\hat{\otimes}_\nabla\mathcal{D}_2\right)\right]$$

where $[({}_A(X)_B,\mathcal{D})]$ denotes the corresponding KK-class and $1\hat{\otimes}_\nabla\mathcal{D}_2$ is short hand for introducing a modified (by a connection) representative of the class in $KK(A,\mathbf{C})$.

This is currently being applied in many situations and I will use it in the next lecture.

Real spectral flow: new idea

In work in progress with John Phillips and Hermann Schulz-Baldes (motivated by condensed matter theory) we have been looking at spectral flow in the space of real skew-adjoint Fredholm operators.

The latter is the real equivalent of the space of self adjoint Fredholm operators on a complex Hilbert space.

The pairing here is

$$KKO^1(\mathbb{R}, \mathcal{A}) \times KKO^1(\mathcal{A}, \mathbb{R}) \to KKO^2(\mathbb{R}, \mathbb{R}) \cong \mathbb{Z}_2$$

The puzzle is the unbounded version of this (bounded seems OK, see paper to appear on arXiv).

Comments on theoretical models in condensed matter physics

Big literature: see our forthcoming review article.
(Atland-Zirnbauer, Kitaev, Freed-Moore........)

Our objective: use KK and KKO to resolve some questions about models in d=2 and d=3 dimensions.

Crystal structure represented by $\mathbb{Z}^d \subset \mathbb{R}^d$.

Magnetic Schrödinger operators: Bloch theory (eg Hall effect).

The Hall conductance: the work of Bellissard.

Translation invariance when there is a magnetic field is restored using projective representations (magnetic translations) and this leads to the irrational rotation algebra playing a role.

Dirac type operators also enter (taking account of spin) so we have chiral symmetry.

Hilbert spaces for continuum models $L^2(\mathbb{R}^d) \otimes \mathbb{C}^N$. where $\mathbb{C}^N$ represents additional internal degrees of freedom

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7. 21–25 Nov. Classification of C*-algebras
   G. Elliott, K. R. Strung, W. Winter, J. Zacharias

Contemporary interest is in distinguishing different topological phases in theoretical models of conducting or insulating materials.

These arise in the study of topological insulators such as in the Kane-Mele model.

It is the first model of this kind in which a $\mathbf{Z}_2$ valued invariant distinguishing different phases arose.

I should mention however that long ago John Lewis and collaborators used the Atiyah-Singer mod 2 index to distinguish phases in the two dimensional Ising model. That is they used KO-theory.

Previous approaches to these models that use the Chern character will not detect such an invariant.

Our idea is to use KKO for this purpose as Kane-Mele incorporates a conjugate linear time reversal symmetry of the Hamiltonian.

We always work in the tight binding approximation that is with a discrete model on $L^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$.

New phenomena have to be modelled: insulating interior but with boundary currents.

Then we have the question of the relation between bulk and edge.

NB: These models are not fundamental, they are just effective theories that capture a part of what is going on in real materials.

But: these phenomena are seen in recently discovered materials so a situation where K-theory predicts physics.

Example: the integer quantum Hall effect: a test case

(Bellissard, Schulz-Baldes).

Main simplifying idea: replace continuum model by a lattice model, differential operators by bounded operators.

(Mathematically, P. McCann showed in the 90s that for the Hall effect example this amounts to studying a Morita equivalent situation)

In the tight binding model of the integer quantum Hall effect without boundary we work on $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ or $\ell(\mathbb{Z}^2)$.

There are two sets of operators:
(i) the symmetries and
(ii) the operators needed to define the Hamiltonian.

The symmetries are called the magnetic translations and we denote them by $\widehat{U}$ and $\widehat{V}$ (unitary operators on $\ell^2(\mathbf{Z}^2)$).

These operators commute with the unitaries U and V that generate the Hamiltonian $H = U + U^* + V + V^*$.

To write them down we choose the 'Landau gauge' such that

$$\begin{aligned}
(\widehat{U}\lambda)(m,n) &= \lambda(m-1,n), & (\widehat{V}\lambda)(m,n) &= e^{-2\pi i\phi m}\lambda(m,n-1), \\
(U\lambda)(m,n) &= e^{-2\pi i\phi n}\lambda(m-1,n), & (V\lambda)(m,n) &= \lambda(m,n-1),
\end{aligned}$$

where ϕ has the interpretation as the magnetic flux through a unit cell of the lattice $\mathbb{Z}^2$ and $\lambda \in \ell^2(\mathbf{Z}^2)$.

The introduction of the phase factor in these formulas is what replaces the connection in the continuum model.

We note the following relations:

$$\widehat{U}\widehat{V} = e^{2\pi i\phi}\widehat{V}\widehat{U} \text{ and } UV = e^{-2\pi i\phi}VU,$$

so $C^*(\widehat{U},\widehat{V}) \cong \mathcal{A}_\phi$ and $C^*(U,V) \cong \mathcal{A}_{-\phi}$.

That is, we have two irrational rotation algebras.

We can also interpret $\mathcal{A}_{-\phi} \cong \mathcal{A}_\phi^{\rm op}$, where $\mathcal{A}^{\rm op}$ is the opposite algebra with multiplication $(ab)^{\rm op} = b^{\rm op}a^{\rm op}$.

Our choice of Landau gauge also means that $C^*(\widehat{U},\widehat{V})$ has the structure of a crossed product C^*-algebra.

It is isomorphic to $C^*(\widehat{U}) \rtimes_\eta \mathbf{Z}$, where $\widehat{V}$ is implementing the crossed-product structure via the automorphism $\eta(\widehat{U}^m) = \widehat{V}^*\widehat{U}^m\widehat{V}$.

The spectral triple

Introduce $(X_1 \pm iX_2)\lambda(m,n) = (m \pm in)\lambda(m,n)$ for $\lambda \in \ell^2(\mathbf{Z}^2)$ that lie in the domain of this operator.

We can calculate the commutation relations of $X_1 \pm iX_2$ with the generating unitaries and see that the result is bounded.

The structure of this example can be viewed another way in terms of projective representations of groups.

In this viewpoint we let $\sigma(k, k') = e^{2\pi i \phi k_1' k_2}$ be a group 2-cocycle for $\mathbf{Z}^2$.

Then $C^*(U, V)$ gives the right regular σ-representation of $\mathbf{Z}^2$ and there is a corresponding left regular $\overline{\sigma}$-representation of $\mathbf{Z}^2$ by $C^*(\widehat{U}, \widehat{V})$.

These each generate the commutant of the other.

Because $C^*(U,V) \cong \mathcal{A}_{-\phi} \cong \mathcal{A}_\phi^{\mathrm{op}}$, we obtain the following.

The data

$$\left(\mathcal{A}_\phi \otimes \mathcal{A}_\phi^{\mathrm{op}}, \ell^2(\mathbf{Z}^2) \oplus \ell^2(\mathbf{Z}^2), \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

defines an even spectral triple.

The Fermi projector

In many, but not all, experimentally observed materials most of the electrons are bound to the constituent atoms. Conduction occurs only if there is a ‘conduction band’ that is, an allowed set of energy levels in the conductor where electrons (with sufficient energy) can reside in an unbound state. This is usually separated by an energy gap from the bound electrons. If the gap is very large electrons will not reach the conducting band just by thermal excitations. It can happen that all energy levels below the gap are already filled with electrons and that there are some left over electrons that occupy this conduction band.

Mathematically speaking, the Fermi projection is a spectral projector for the Hamiltonian that describes the electron motion in the conductor. It projects onto the bound states of the electrons.

There has to be a conduction band in the complement of the range of the Fermi projector for materials to behave as conductors. Otherwise they are insulators.

Kasparov theory and the bulk-edge correspondence

Recall that we have magnetic translations $\widehat{U}$ and $\widehat{V}$ as unitary operators on $\ell^2(\mathbf{Z}^2)$.

These operators commute with the unitaries U and V that generate the Hamiltonian $H = U + U^* + V + V^*$, where

$$(\widehat{U}\lambda)(m,n) = \lambda(m-1,n), \qquad (\widehat{V}\lambda)(m,n) = e^{-2\pi i\phi m}\lambda(m,n-1),$$
$$(U\lambda)(m,n) = e^{-2\pi i\phi n}\lambda(m-1,n), \qquad (V\lambda)(m,n) = \lambda(m,n-1),$$

We would also like to consider a system with boundary. This uses the Hilbert space $\ell^2(\mathbf{Z} \times \mathbf{N})$.

The bulk-edge correspondence is about linking the topological properties of the 'bulk' (boundary-free) system to a system with an edge.

Let S be the unilateral shift operator on $\ell^2(\mathbf{N})$ with $S^*S = 1$, $SS^* = 1 - P_{n=0}$.

We use the notation $\mathcal{K}(\mathcal{H})$ to denote the compact operators on a Hilbert space $\mathcal{H}$.

Kellendonk, Richter and Schulz-Baldes link 'bulk' (no boundary) and edge systems via the short exact sequence

$$0 \to C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbf{N})] \xrightarrow{\psi} C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S) \to C^*(\widehat{U}, \widehat{V}) \to 0,$$

where ψ on generators is given by

$$\psi(\widehat{U}^m \otimes e_{jk}) = (\widehat{V}^*)^j \widehat{U}^m \widehat{V}^k \otimes S^j P_{n=0} (S^*)^k$$

for matrix units e_{jk} in $\mathcal{K}[\ell^2(\mathbf{N})]$ and then extended to the full algebra by linearity.

In this sequence the quotient algebra $C^*(\widehat{U}, \widehat{V})$ is the one applicable to the bulk (i.e. no boundary) system.

The algebra $C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbf{N})]$ is an ideal in $C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$.

We think of the ideal as operators which act on $\ell^2(\mathbf{Z} \times \mathbf{N})$ and decay sufficiently fast away from the edge $\mathbf{Z} \times \{0\}$.

Building a Kasparov module

Abstract theory tells us that short exact sequences of the type

$$0 \to C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbf{N})] \to C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S) \to A_\phi \to 0$$

give rise to a class in $KK^1(A_\phi, C^*(\widehat{U}))$. Let's make this explicit using unbounded Kasparov theory.

We have to build an explicit representative of the class in $KK^1(A_\phi, C^*(\widehat{U}))$.

First we need a $C^*(\widehat{U})$-valued inner product on $C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$.

Basically the idea for what follows came from studying exact sequences of Cuntz-Pimsner algebras. Here it reduces to working with the generators and their relations to get what we need.

To begin, we hypothesise the existence of Ψ, some linear functional (ie complex valued) on $C^*(S) \subset C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$ and consider the formula

$$\begin{aligned}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1} \otimes S^{l_1}(S^*)^{l_2}\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2} \otimes S^{n_1}(S^*)^{n_2}\Big) \\ := \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\Big)^* \widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\,\Psi\Big[\Big(S^{l_1}(S^*)^{l_2}\Big)^* S^{n_1}(S^*)^{n_2}\Big] \\ = \widehat{U}^{-m_1}\widehat{V}^{n_1-n_2-(l_1-l_2)}\widehat{U}^{m_2}\,\Psi\Big[S^{l_2}(S^*)^{l_1}S^{n_1}(S^*)^{n_2}\Big]\end{aligned}$$

for $m_1, m_2 \in \mathbf{Z}$ and $n_1, n_2, l_1, l_2 \in \mathbf{N}$.

By an appropriate choice of Ψ this will become the inner product

Building a Kasparov module (cont.)

After working with this for a while we realised we wanted the functional Ψ to have the property that $\Psi\big[S^{l_2}(S^*)^{l_1}S^{n_1}(S^*)^{n_2}\big] = \delta_{l_1-l_2,n_1-n_2}$ and that $\Psi(T)=0$ if T is compact.

The functional

$$\Psi(T) = \operatorname*{res}_{s=1} \sum_{k=0}^{\infty} \langle e_k, Te_k\rangle (1+k^2)^{-s/2}$$

does the trick, where $\{e_k\}$ is any basis of $\ell^2(\mathbf{N})$.

Therefore the formula on the previous slide becomes:

$$\begin{aligned}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big) \\ = \widehat{U}^{-m_1}\widehat{V}^{n_1-n_2-(l_1-l_2)}\widehat{U}^{m_2}\,\delta_{l_1-l_2,n_1-n_2} \\ = \widehat{U}^{m_2-m_1}\delta_{l_1-l_2,n_1-n_2}\end{aligned}$$

so we have a $C^*(\widehat{U})$-valued inner-product.

We also introduce the right-action of $C^*(\widehat{U})$, where for any $\alpha \in \mathbf{Z}$,

$$\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) \cdot \widehat{U}^\alpha = \widehat{V}^{n_1-n_2}\widehat{U}^{m+\alpha} \otimes S^{n_1}(S^*)^{n_2}.$$

We divide out the zero-length vectors of $C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$ in the norm induced by $(\cdot \mid \cdot)_{C^*(\widehat{U})}$ (non-trivial!) and complete to obtain the C^*-module $Z_{C^*(\widehat{U})}$.

Next, we need an adjointable left-action by $\mathcal{A}_\phi \cong C^*(\widehat{U}, \widehat{V})$. We define on generating elements

$$\begin{aligned}(\widehat{U}^\alpha\widehat{V}^\beta) \cdot \left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) \\ &= (\widehat{U}^\alpha\widehat{V}^\beta\widehat{V}^{n_1-n_2}\widehat{U}^m) \otimes S^{n_1+\beta}(S^*)^{n_2} \\ &= e^{2\pi i\phi\alpha(n_1-n_2+\beta)}\widehat{V}^{\beta+n_1-n_2}\widehat{U}^{m+\alpha} \otimes S^{\beta+n_1}(S^*)^{n_2}\end{aligned}$$

for $\alpha, \beta \in \mathbf{Z}$ with $\beta \geq 0$ and an analogous formula but with $S^{n_1}(S^*)^{n_2+|\beta|}$ for $\beta < 0$.

An unenlightening computation shows that this representation is adjointable under the inner product $(\cdot \mid \cdot)_{C^*(\widehat{U})}$.

Finally, we need for an unbounded Kasparov module the unbounded operator of the definition.

Introduce the 'number operator' $N : \mathrm{Dom}(N) \subset Z_{C^*(\widehat{U})} \to Z_{C^*(\widehat{U})}$ on generating elements as

$$N\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) = (n_1 - n_2)\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}.$$

Proposition

The triple $\left({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N\right)$ is an odd unbounded Kasparov module. Furthermore, the corresponding class $\left[({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N)\right]$ in $KK^1(A_\phi, C^*(\widehat{U}))$ is the same as the class induced by the short-exact sequence linking the bulk and edge algebras.

Proof: A calculation.

It relies on the singular nature of the inner-product and the functional Ψ.

Note that the idea for defining Ψ in this way came from a long exposure to the theory of Dixmier traces as expounded in Alain Connes' book (Noncommutative geometry).

Edge spectral triple

Next we consider our edge algebra $C^*(\widehat{U})$ acting as shift operators on the space $\ell^2(\mathbf{Z})$.

We have a natural spectral triple in this setting given by

$$\left(C^*(\widehat{U}), \ell^2(\mathbf{Z}), M\right),$$

where $M : \mathrm{Dom}(M) \to \ell^2(\mathbf{Z})$ is given by $M\lambda(m) = m\lambda(m)$.

Our spectral triple is an odd unbounded $C^*(\widehat{U})$-$\mathbf{C}$ Kasparov module and so gives a class in $KK^1(C^*(\widehat{U}), \mathbf{C})$.

Factorisation of bulk triple

To review, we have our bulk triple giving a class in $KK^0(A_\phi, \mathbf{C})$, the Kasparov module representing the short exact sequence giving a class in $KK^1(A_\phi, C^*(\widehat{U}))$ and an edge spectral triple giving a class in $KK^1(C^*(\widehat{U}), \mathbf{C})$.

Theorem

*(**Factorisation**) ([Bourne-C-Rennie] Under the internal Kasparov product*

$$KK^1(A_\phi, C^*(\widehat{U})) \times KK^1(C^*(\widehat{U}), \mathbf{C}) \to KK^0(A_\phi, \mathbf{C})$$

we have that

$$\left[({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N)\right] \hat{\otimes}_{C^*(\widehat{U})} \left[({}_{C^*(\widehat{U})}(\ell^2(\mathbf{Z}))_{\mathbf{C}}, M)\right] = -\left[({}_{A_\phi}(\ell^2(\mathbf{Z}^2))_{\mathbf{C}}, X, \gamma)\right]$$

where $-[X]$ denotes the inverse class in the KK-group.

The proof of this Theorem is a long calculation. It relies on the explicit formula for the Kasparov product that the unbounded picture gives us.

Pairings and the bulk-edge correspondence (QHE)

Recall that Bellissard's expression for the Hall conductance comes from the pairing of the K-theory class of the Fermi projection $[P_\mu] \in KK(\mathbf{C}, A_\phi)$ with the bulk spectral triple; that is,

$$\sigma_H = \frac{e^2}{h} \left([P_\mu] \,\hat{\otimes}_{A_\phi} \left[({}_{A_\phi}(\ell^2(\mathbf{Z}^2))_{\mathbf{C}}, X, \gamma) \right] \right).$$

We can now use the Theorem to rewrite $({}_{A_\phi}(\ell^2(\mathbf{Z}^2))_{\mathbf{C}}, X, \gamma)$ as a Kasparov product and then we obtain:

$$\sigma_H = -\frac{e^2}{h} \left([P_\mu] \,\hat{\otimes}_{A_\phi} \left[({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N) \right] \hat{\otimes}_{C^*(\widehat{U})} \left[({}_{C^*(\widehat{U})}(\ell^2(\mathbf{Z}))_{\mathbf{C}}, M) \right] \right).$$

The bulk-edge correspondence now follows from the associativity of the Kasparov product.

To see this, note that our proposed ‘edge conductance’ is in

$$-\frac{e^2}{h}\left([P_\mu]\,\hat{\otimes}_{A_\phi}\left[({}_{A_\phi}(Z)_{C^*(\widehat{U})}, N)\right]\right)\hat{\otimes}_{C^*(\widehat{U})}\left[({}_{C^*(\widehat{U})}(\ell^2(\mathbf{Z}))_{\mathbf{C}}, M)\right],$$

The first tensor product is composing an element $[P_\mu] \in KK^0(\mathbf{C}, A_\phi)$ and an element of $KK^1(A_\phi, C^*(\widehat{U}))$ and hence gives an element in $KK^1(\mathbf{C}, C^*(\widehat{U}))$.

Thus the second tensor product composes an element of $KK^1(\mathbf{C}, C^*(\widehat{U}))$ with one from $KK^1(C^*(\widehat{U}), \mathbf{C})$, a pairing of the K-theory and K-homology of our edge algebra.

The result is an element of $KK^0(\mathbf{C}, \mathbf{C})$ that coincides with the Bellissard condcutance.

The details may be found in the:

Reference: arXiv:1411.7527, The bulk-edge correspondence for the quantum Hall effect in Kasparov theory,
Chris Bourne, Alan L. Carey, Adam Rennie

1. Freed-Moore and Thiang: Symmetries and Clifford algebras

The symmetry groups of interest are subgroups of the CT-symmetry group of involutions $\{1, T, C, CT\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, where $T \equiv$ time reversal and $C \equiv$ charge conjugation.

In quantum mechanics T and C are implemented by conjugate linear (not complex linear) maps. This led physicists and mathematicians to use KO-theory.

Historically, Atland-Zirnbauer and Kitaev are the main people associated with this KO viewpoint.

Important: this is not the whole story. Space reversal symmetry (parity) and crystallographic symmetries also enter. Parity suggests KKR while other symmetries require equivariant theory.

From a mathematical perspective Freed-Moore and Thiang gave a systematic approach based on Karoubi..

Definition A gapped Hamiltonian H acting on $\mathcal{H}$ is compatible with $\{1, T, C, CT\}$ if there are conjugate-linear operators R_T and R_C on $\mathcal{H}$ such that $R_T^2, R_C^2 \in \{\pm 1_{\mathcal{H}}\}$ and

$$R_T H R_T^* = H, \qquad R_C H R_C^* = -H, \qquad R_{CT} H R_{CT}^* = -H.$$

Proposition [Freed-Moore, Thiang] If H is compatible with a subgroup G of $\{1, T, C, CT\}$, then there is a graded representation of a real or complex Clifford algebra on $\mathcal{H}$ with grading $\Gamma = H|H|^{-1} = \mathrm{sgn}(H)$.

The Clifford algebra is determined precisely by the symmetries present and whether they are even or odd (‘the tenfold way’).

Symmetries and representations

Symmetry generators	R_C^2	R_T^2	Graded Clifford representation (up to stable isomorphism)
T		$+1$	$C\ell_{0,0}$
C, T	$+1$	$+1$	$C\ell_{1,0}$
C	$+1$		$C\ell_{2,0}$
C, T	$+1$	-1	$C\ell_{3,0}$
T		-1	$C\ell_{4,0}$
C, T	-1	-1	$C\ell_{5,0}$
C	-1		$C\ell_{6,0}$
C, T	-1	$+1$	$C\ell_{7,0}$
N/A			$\mathbf{C}\ell_0$
CT	$R_{CT}^2 = 1$		$\mathbf{C}\ell_1$

Table: Symmetry types and their corresponding graded Clifford representations.

3. Why KKO?

We would like to capture the index theory implied by the Freed-Moore-Thiang point of view. Because R_T and R_C are anti-linear, we require real C^*-algebras and KKO-theory.

KKO-theory requires real Hilbert C^*-modules.

Basically the same definition as in the complex case with obvious modifications.

As in the case of Hilbert spaces, we are interested in linear transformations between C^*-modules.

So as before we have $\mathrm{End}_B(E)$ the adjointable endomorphisms on the Hilbert C^*-module E_B.

We need the following fact:

if $\phi : A \to \mathrm{End}_B(E)$ is a $\mathbf{Z}_2$-graded homomorphism for real C^*-algebras A and B, then ϕ determines a class $[\phi]$ in Kasparov's real $KKO(A, B)$ group.

Hamiltonians

Unless otherwise stated, we will assume the Hamiltonians act on $\ell^2(\mathbf{Z}^d) \otimes \mathbb{C}^N$ and are represented by matrices of operators whose entries are finite polynomials of shift operators.

As usual we have to assume Hamiltonians also have a spectral gap containing the Fermi level.

If H is compatible with the symmetry group G, a subgroup of $\{1, T, C, CT\}$, then we also assume that the symmetry action $H \mapsto R_g H R_g^*$ extends to an action on the algebra generated by the shift operators that generate H.

From this we choose our observable algebra A to be the *real* C^*-algebra generated by the shift operators that generate H.

A KKO-class encoding symmetries

Take G a subgroup of $\{1, T, C, CT\}$ and take the crossed product $A \rtimes G$. There is a conditional expectation on the crossed-product, $\Phi : A \rtimes G \to A$ with $\Phi\Big(\sum_g a_g U_g\Big) = a_e$.

Define the right-A C^*-module E_A as the completion of $A \rtimes G$ under the inner-product

$$(e_1 \mid e_2)_A = \Phi(e_1^* e_2)$$

and right-action by multiplication.

As before we assume there is a gap in the spectrum of the Hamiltonian and the Fermi projection $[P_\mu^G]$ projects onto states with energies below the gap.

Now we introduce our KKO representative which we use to classify the gapped Hamiltonians via their symmetries.

Proposition There is a graded homomorphism $\phi : C\ell_{n,0} \to \mathrm{End}_A(E)$ (or, in the complex case, $\phi : \mathbf{C}\ell_n \to \mathrm{End}_A(E)$) graded by $H|H|^{-1} = \mathrm{sgn}(H)$ and with n determined by the symmetry group G.

Therefore we obtain a class $[P_\mu^G]$ in $KKO(C\ell_{n,0}, A)$ or $KK(\mathbf{C}\ell_n, A_{\mathbf{C}})$. We think of $[P_\mu^G]$ as the class of the Fermi projection but taking into account the internal symmetries of the Hamiltonian.

4. Bulk spectral triple

Recall from the integer quantum Hall effect the (complex) spectral triple

$$\left(\mathcal{A}_\phi \otimes \mathbf{C}, \ell^2(\mathbf{Z}^2) \otimes \mathbf{C}^2, \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

For d-dimensional systems with no anti-linear symmetries, we have the higher-dimensional (complex) spectral triple

$$\left(\mathcal{A}_\phi^d \otimes \mathbf{C}, \ell^2(\mathbf{Z}^d) \otimes \mathbf{C}^\nu, \sum_{j=1}^{d} X_j \otimes \gamma^j, \gamma = (-i)^{d/2}\gamma^1 \cdots \gamma^d\right),$$

where $\{\gamma^j\}_{j=1}^d$ are matrices in $M_\nu(\mathbf{C})$ and such that $\gamma^i\gamma^j + \gamma^j\gamma^i = \delta_{i,j}$.

We have to refine this idea to real algebras and spectral triples.

Bulk spectral triple (cont.)

Our observable algebra $A = A_\phi^d$ acts on the real Hilbert space $\mathcal{H}_b \cong \ell^2(\mathbf{Z}^d) \otimes \mathbb{F}^N$, where $\mathbb{F} = \mathbf{R}$ or $\mathbf{C} \cong \mathbf{R} \oplus i\mathbf{R}$. We also have the dense subalgebra of say, polynomials in the twisted shift operators, $\mathcal{A}_\phi^d \subset A_\phi^d$.

Proposition The tuple

$$\lambda = \left(\mathcal{A}_\phi^d \hat{\otimes} C\ell_{0,d},\, \mathcal{H}_b \otimes \bigwedge\nolimits^* \mathbf{R}^d,\, \mathcal{D} = \sum_{j=1}^{d} X_j \otimes \gamma^j,\, \gamma_{\bigwedge^* \mathbf{R}^d} \right)$$

is a real spectral triple.

The left-action of $C\ell_{0,d}$ is generated by the operators $\{\rho^j\}_{j=1}^d$ and the operators $\{\gamma^j\}_{j=1}^d$ generate $C\ell_{d,0}$. The Clifford algebras $C\ell_{0,d}$ and $C\ell_{d,0}$ are represented as left and right actions on $\bigwedge^* \mathbf{R}^d$ respectively by the formulae

$$\rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega, \qquad \gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega,$$

with $\omega \in \bigwedge^* \mathbf{R}^d$ and $\{e_j\}_{j=1}^d$ the standard basis of $\mathbf{R}^d$.

5. The Kasparov product

Again we need the intersection product

$$KKO(A,B) \times KKO(B,C) \to KKO(A,C)$$

for separable and nuclear real C^*-algebras A, B, and C.

By working in unbounded Kasparov theory (spectral triples etc.) the Kasparov product can be computed in a more constructive manner. Then we can obtain $\mathbb{Z}_2$ invariants in an explicit way.

6. The index pairing in KKO

We have the class $[P_\mu^G] \in KKO(C\ell_{n,0}, C^*_\phi(\mathbf{Z}^d))$.

The bulk spectral triple gives a class $[\lambda]$ in $KKO(C^*_\phi(\mathbf{Z}^d)\hat{\otimes}C\ell_{0,d}, \mathbf{R})$.

Thus using the fact that we can 'move Clifford algebras' there is a well-defined map

$$KKO(C\ell_{n,0}, A) \times KKO(A\hat{\otimes}C\ell_{0,d}, \mathbf{R}) \to KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,d}, \mathbf{R})$$

and this leads to a Clifford module valued index (in the sense of Atiyah-Bott-Shapiro):

$$C_{n,d} = [P_\mu^G]\hat{\otimes}_A[\lambda] \in KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,d}, \mathbf{R}) \cong KO_{n-d}(\mathbf{R})$$

By taking the index pairing/Kasparov product of these classes, we obtain topological information of interest in the system.

Pairings and the periodic table

Symmetry generators	R_C^2	R_T^2	Graded Rep.	$[P_\mu^G]\hat{\otimes}[\lambda] \in KO_{n-d}(\mathbf{R})$ or $K_{n-d}(\mathbf{C})$ $d=0$	$d=1$	$d=2$	$d=3$
T		$+1$	$C\ell_{0,0}$	$\mathbf{Z}$	0	0	0
C,T	$+1$	$+1$	$C\ell_{1,0}$	$\mathbf{Z}_2$	$\mathbf{Z}$	0	0
C	$+1$		$C\ell_{2,0}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$	0
C,T	$+1$	-1	$C\ell_{3,0}$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$
T		-1	$C\ell_{4,0}$	$2\mathbf{Z}$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$
C,T	-1	-1	$C\ell_{5,0}$	0	$2\mathbf{Z}$	0	$\mathbf{Z}_2$
C	-1		$C\ell_{6,0}$	0	0	$2\mathbf{Z}$	0
C,T	-1	$+1$	$C\ell_{7,0}$	0	0	0	$2\mathbf{Z}$
N/A			$\mathbf{C}\ell_0$	$\mathbf{Z}$	0	$\mathbf{Z}$	0
CT	$R_{CT}^2 = 1$		$\mathbf{C}\ell_1$	0	$\mathbf{Z}$	0	$\mathbf{Z}$

Table: Symmetry types, their corresponding graded Clifford representation and the pairing of the Fermi projection with the d-dimensional spectral triple (shown for $d \leq 3$).

7. Clifford module valued index and Atiyah-Bott-Shapiro

The $2\mathbf{Z}$ in the previous table arises from the quaternionic index of Atiyah-Singer which we now explain.

Recall the product

$$C_{n,d} = [P_\mu^G]\hat{\otimes}_A[\lambda] \in KO_{n-d}(\mathbf{R}).$$

The class $C_{n,d}$ is represented by the spectral triple

$$\Big(C\ell_{n,0}\hat{\otimes}C\ell_{0,d},\, (E^N \otimes_A \mathcal{H}_b) \otimes \bigwedge^* \mathbf{R}^d,\, \widetilde{X},\, (\mathrm{sgn}(H) \otimes 1)\hat{\otimes}\gamma_{\bigwedge^* \mathbf{R}^d}\Big).$$

Under the $\mathbf{Z}_2$-grading, we can express $\widetilde{X} = \begin{pmatrix} 0 & \widetilde{X}_- \\ \widetilde{X}_+ & 0 \end{pmatrix}$, where $\widetilde{X}_\pm$ are real elliptic operators. The operator $\widetilde{X}$ (graded) commutes with a left action of $C\ell_{n,0}\hat{\otimes}C\ell_{0,d} \cong C\ell_{n,d}$.

By ellipticity, we may say that $\mathrm{Ker}(\widetilde{X}) \cong \mathrm{Ker}(\widetilde{X})^0 \oplus \mathrm{Ker}(\widetilde{X})^1$ is a $\mathbf{Z}_2$-graded $C\ell_{n,d}$-module. Furthermore, $\mathrm{Ker}(\widetilde{X})^0 \cong \mathrm{Ker}(\widetilde{X}_+)$.

Clifford index (cont.)

Definition [Atiyah-Bott-Shapiro] Denote by $\hat{\mathfrak{M}}_{r,s}$ the Grothendieck group of equivalence classes of real $\mathbf{Z}_2$-graded modules with an irreducible graded left-representation of $C\ell_{r,s}$.

$\mathrm{Ker}(\widetilde{X})$ determines a class in the quotient group $\hat{\mathfrak{M}}_{n,d}/i^*\hat{\mathfrak{M}}_{n,d+1}$, where i^* comes from restricting a Clifford action of $C\ell_{n,d+1}$ to $C\ell_{n,d}$. Next, we use the Atiyah-Bott-Shapiro isomorphism to relate

$$\hat{\mathfrak{M}}_{n,d}/i^*\hat{\mathfrak{M}}_{n,d+1} \cong KO_{n-d}(\mathbf{R}).$$

Definition The Clifford index of $\widetilde{X}$ is given by

$$\mathrm{Index}_{n-d}(\widetilde{X}) := [\mathrm{Ker}(\widetilde{X})] \in \hat{\mathfrak{M}}_{n,d}/i^*\hat{\mathfrak{M}}_{n,d+1} \cong KO_{n-d}(\mathbf{R}).$$

Index_k is a generalisation of the Fredholm index.

8. Kane-Mele as an example

We take a Hamiltonian on $\ell^2(\mathbf{Z}^2) \otimes \mathbf{C}^N$ that is compatible with the group $\{1, T\}$ and $R_T^2 = -1$. Our observable algebra is $C^*(\mathbf{Z}^2)$ acting as matrices of shift operators.

By our general method, we obtain a class $[P_\mu^G] \in KKO(C\ell_{4,0}, C^*(\mathbf{Z}^2))$.

We can also build the spectral triple

$$\lambda_{KM} = \left(\mathcal{A}\hat{\otimes}C\ell_{0,2}, \ell^2(\mathbf{Z}^2) \otimes \mathbf{C}^{2N} \otimes \bigwedge\nolimits^{*} \mathbf{R}^2, X, \gamma\right),$$

where under a specific choice of Clifford generators

$$X = \begin{pmatrix} 0_{2N} & X_1 \otimes 1_{2N} - iX_2 \otimes 1_{2N} \\ X_1 \otimes 1_{2N} + iX_2 \otimes 1_{2N} & 0_{2N} \end{pmatrix}.$$

Kane-Mele (cont.)

Our index pairing gives a map

$$KKO(C\ell_{4,0}, C^*(\mathbf{Z}^2)) \times KKO(C^*(\mathbf{Z}^2)\hat{\otimes} C\ell_{0,2}, \mathbf{C}) \to KO_2(\mathbf{R})$$
$$\big([P_\mu^G], [X]\big) \mapsto \mathrm{Index}_{4-2}(\widetilde{X}) \in KO_2(\mathbf{R}) \cong \mathbf{Z}_2$$

and so we obtain the $\mathbf{Z}_2$ invariant of interest.