Quantum isometry of classical and noncommutative spaces

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Based on joint works with many students and collaborators during 2009-15: Bhowmick, Skalski, Das, Mandal, Joardar, Banica, Etingof and Walton.

Introduction and motivation

- Symmetry is very important to understand any physical model and often it paves the way for solving the equations involved.
- Classical concept: Symmetry given by group actions.
- Generalization : groups replaced by quantum groups.
- So, it is natural to conceive of 'universal quantum symmetry', or 'quantum automorphism group' of some mathematical structure.
- Manin formulated in purely Hopf algebraic terms.
- Motvated by Connes, S. Wang came up with a version in the world of C^* algebraic (compact) quantum group.
- My own motivation: extend the philosophy of quantum automorphism to geometry, both commutative and noncommutative, by formulating the notion of quantum isometry.

Quick review of basic concepts

Definition

a **compact quantum group** *(CQG for short) a la Woronowicz is a pair* $(\mathcal{A}, \Delta)$ *where* $\mathcal{A}$ *is a unital* C^**-algebra,* Δ *is a coassociative comultiplication, i.e. a unital* C^**-homomorphism from* $\mathcal{A}$ *to* $\mathcal{A} \otimes \mathcal{A}$ *(minimal tensor product) satisfying* $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$*, and linear span of each of the sets* $\{(b \otimes 1)\Delta(c) \colon b, c \in \mathcal{A}\}$ *and* $\{(1 \otimes b)\Delta(c) \colon b, c \in \mathcal{A}\}$ *is dense in* $\mathcal{A} \otimes \mathcal{A}$

There is a natural generalisation of group action on spaces in this noncommutative set-up, which is given below :

Definition

We say that a CQG $(\mathcal{A}, \Delta)$ *acts on a (unital)* C^**-algebra* $\mathcal{C}$ *if there is a unital* $*$*-homomorphism* $\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}$ *such that* $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$*, and the linear span of* $\alpha(\mathcal{C})(1 \otimes \mathcal{A})$ *is norm-dense in* $\mathcal{C} \otimes \mathcal{A}$*.*

Noncommutative geometry a la Connes

Definition

A **spectral triple** *or* **spectral data** *is a tuple* $(\mathcal{A}, \mathcal{H}, D)$ *where* $\mathcal{H}$ *is a separable Hilbert space,* $\mathcal{A}$ *is a* $*$*-subalgebra of* $\mathcal{B}(\mathcal{H})$ *(not necessarily norm-closed) and* D *is a self-adjoint (typically unbounded) operator such that for each* $a \in \mathcal{A}$*, the operator* $[D, a]$ *admits bounded extension. Such a spectral triple is also called an* **odd spectral triple***. If in addition, we have* $\gamma \in \mathcal{B}(\mathcal{H})$ *satisfying* $\gamma = \gamma^* = \gamma^{-1}$*,* $D\gamma = -\gamma D$ *and* $[a, \gamma] = 0$ *for all* $a \in \mathcal{A}$*, then we say that the quadruplet* $(\mathcal{A}, \mathcal{H}, D, \gamma)$ *is an* **even spectral triple** *or* **even spectral data***. The operator* D *is called the* **Dirac operator** *corresponding to the spectral triple. We say that the spectral triple is of* **compact type** *if* D *has compact resolvents. It is* Θ*-summable if* $Tr(e^{-tD^2}) < \infty$ *for* $t > 0$*.*

Background

- Early work : formulation of quantum automorphism and quantum permutation groups by Wang, and follow-up work by Banica, Bichon and others.
- Basic principle: For some given mathematical structure (e.g., a finite set, a graph, a C^* or von Neumann algebra) identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type.
- However, most of the earlier work done concerned some kind of quantum automorphism groups of a 'finite' structure. So, one should extend these to the 'continuous'/ 'geometric' set-up. This motivated my definition of quantum isometry group in [3].

Wang's quantum permutation and quantum automorphism groups

Quantum permutation group (Wang):
Let $X = \{1, 2, ..., n\}$, G group of permutations of X. G can be identified as the universal object in the category of groups acting on X. For a simiar (bigger) category of compact quantum groups acting on $C(X)$, Wang obtained the following universal object:

$$\mathcal{Q} := C^* \left(q_{ij}, i, j = 1, ..., n; \mid q_{ij} = q_{ij}^* = q_{ij}^2, \sum_i q_{ij} = 1 = \sum_j q_{ij}\right.$$

The co product is given by $\Delta(q_{ij}) = \sum_k q_{ik} \otimes q_{kj}$, and the action on $C(X)$ is given by $\alpha(\chi_i) = \sum_j \chi_j \otimes q_{ji}$.
This CQG is naturally called 'quantum permutation group' of n objects.

However, the cateogory of CQG acting on M_n does NOT have a universal object!
Remedy (due to Wang): consider the subcategory of actions which preserves a given faithful state.
More precisely: For an $n \times n$ positive invertible matrix $Q = (Q_{ij})$, let $A_u(Q)$ be the universal C^*-algebra generated by $\{u_{kj}, k, j = 1, ..., d_i\}$ such that $u := ((u_{kj}))$ satisfies

$$uu^* = I_n = u^* u, \quad u' Q \overline{u} Q^{-1} = I_n = Q \overline{u} Q^{-1} u'.$$

Here $u' = ((u_{ji}))$ and $\overline{u} = ((u_{ij}^*))$. Coproduct given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

Proposition

$A_u(Q)$ is the universal object in the category of CQG which admit a unitary representation, say U, on the finite dimensional Hilbert space $\mathbb{C}^n$ such that ad_U preserves the functional $M_n \ni x \mapsto \mathrm{Tr}(\mathrm{Q}^{\mathrm{T}}\mathrm{x})$.

Quantum isometry in terms of 'Laplacian' (Goswami 2009)

- Classical ismoetries: the group of Riemannian isometries of a compact Riemannian manifold M is the universal object in the category of all compact metrizable groups acting on M, with smooth and isometric action.
- Moreover, a smooth map γ on M is a Riemannian isometry if and only if the induced map $f \mapsto f \circ \gamma$ on $C^\infty(M)$ commutes with the Laplacian $-d^*d$.

Under reasonable regularity conditions on a (compact type, Θ-summable) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, one has analogues of Hilbert space of forms $\mathcal{H}_i^D$, say, $i = 0, 1, ...$ The map $d(a) := [D, a]$ then extends to a (closable, densely defined) map from $\mathcal{H}_0^D$ (space of 0-forms) to $\mathcal{H}_1^D$ (space of one-forms). The self-adjoint negative map $-d^*d$ is the noncommutative analogue of Laplacian $\mathcal{L} \equiv \mathcal{L}_D$, and we additionally assume that

(a) $\mathcal{L}$ maps $\mathcal{A}^\infty$ into itself;
(b) $\mathcal{L}$ has compact resolvents and its eigenvectors belong to $\mathcal{A}^\infty$, forming a norm-total subset of $\mathcal{A}$;
(c) the kernel of $\mathcal{L}$ is one dimensional ("connectedness"). It is then natural to call an action α of some CQG $\mathcal{Q}$ on the C^*-completion of $\mathcal{A}^\infty$ to be *smooth and isometric* if for every bounded linear functional ϕ on $\mathcal{Q}$, one has $(\mathrm{id} \otimes \phi) \circ \alpha$ maps $\mathcal{A}^\infty$ into itself and commutes with $\mathcal{L}$.

Theorem

Under assumptions (a)-(c), there exists a universal object (denoted by $QISO^{\mathcal{L}}$) in the category of CQG acting smoothly and isometrically on the given spectral triple.

- The assumption (c) can be relaxed for classical spectral triples and their Rieffel-deformations.

Quantum isometry in terms of the Dirac operator (Bhowmick-Goswami 2009)

- From the NCG perspective, it is more appropriate to have a formulation in terms of the Dirac operator directly.
- Classical fact: an action by a compact group G on a Riemannian spin manifold is an orientation-preserving isometry if and only if lifts to a unitary representation of a 2-covering group of G on the Hilbert space of square integrable spinors which commutes with the Dirac operator.

For a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type, it is thus reasonable to consider a category $\mathbf{Q}'$ of CQG $(\mathcal{Q}, \Delta)$ having unitary (co-) representation, say U, on $\mathcal{H}$, (i.e. U is a unitary in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{Q})$ such that $(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}$) which commutes with $D \otimes 1_{\mathcal{Q}}$, and for every bounded functional ϕ on $\mathcal{Q}$, $(\mathrm{id} \otimes \phi) \circ \mathrm{ad}_U$ maps $\mathcal{A}^\infty$ into its weak closure. Objects of this category will be called 'orientation preserving quantum isometries'.

If $\mathbf{Q}'$ has a universal object, we denote it by $\widetilde{QISO^+}(D)$. In general, however, $\mathbf{Q}'$ may fail to have a universal object. We do get a universal object in suitable subcategories by fixing a 'volume form'...

Theorem

Let R be a positive, possibly unbounded, operator on $\mathcal{H}$ commuting with D and consider the functional (defined on a weakly dense domain) $\tau_R(x) = Tr(Rx)$. Then there is a universal object (denoted by $\widetilde{QISO^+}_R(D)$) in the subcategory of $\mathbf{Q}'$ consisting of those $(\mathcal{Q}, \Delta, U)$, for which $(\tau_R \otimes \mathrm{id})(\mathrm{ad}_U(\cdot)) = \tau_R(\cdot)1_{\mathcal{Q}}$.

Given such a choice of R, we shall call the spectral triple to be R-twisted.

- The C^*-subalgebra $QISO_R^+(D)$ of $\widetilde{QISO}_R^+(D)$ generated by elements of the form $\{< (\xi \otimes 1), \mathrm{ad}_{U_0}(a)(\eta \otimes 1) >, \ a \in \mathcal{A}^\infty, \ \xi, \eta \in \mathcal{H}\}$, where U_0 is the unitary representation of $\widetilde{QISO}^+{}_R(D)$ on $\mathcal{H}$ and $< \cdot, \cdot >$ denotes the $\widetilde{QISO}_R^+(D)$-valued inner product of the Hilbert module $\mathcal{H} \otimes \widetilde{QISO}_R^+(D)$, will be called the quantum group of orientation and (R-twisted) volume preserving isometries. A similar C^*-subalgebra of $\widetilde{QISO}^+(D)$, if it exists, will be denoted by $QISO^+(D)$.
- However, $QISO_R^+(D)$ may not admit a C^* action for general noncommutative manifolds (but does so for classical manifolds and their Rieffel-deformations at least).
- Under mild conditions $QISO^{\mathcal{L}} \cong QISO_I^+(d + d^*)$. where $d + d^*$ is the 'Hodge Dirac operator' on the space of all (noncommutative) forms.

Computational techniques

- Bhowick, Goswami, Joardar: $QISO^{\mathcal{L}}$ and $QISO_R^+$ of deformed or cocycle-twisted spectral triples is isomorphic with a similar deformed or twisted version of the $QISO^{\mathcal{L}}$ or $QISO_R^+$ of the original (undeformed) spectral triple.
- So, quantum isometries of noncommutative examples obtained from classical manifolds can be computed provided the quantum isometry groups of classical manifolds are known.
- Bhowmick, Goswami, Skalski: The functors $QISO^+$, $QISO_R^+$ etc. commute with inductive limit under suitable conditions, which facilitates computations of quantum isometry groups for spectral triples on AF algebras.
- Other computations include $QISO$ of group algebras (Banica, Skalski, Bhowmick, Soltan and others), free or half-liberated models (Banica, Goswami and others) etc.

QISO for spectral triples with real structures (Goswami 2010)

- A real structure for an odd spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is given by a (possibly unbounded, invertible) closed anti-linear operator $\tilde{J}$ on $\mathcal{H}$ such that $\mathcal{D} := \mathrm{Dom}(D) \subseteq \mathrm{Dom}(\tilde{J})$, $\tilde{J}\mathcal{D} \subseteq \mathcal{D}$, $\tilde{J}$ commutes with D on $\mathcal{D}$, and the antilinear isometry J obtained from the polar decomposition of $\tilde{J}$ satisfies $J^2 = \epsilon I$, $JD = \epsilon' DJ$, and for all $x, y \in \mathcal{A}^\infty$, the commutators $[x, JyJ^{-1}]$ and $[JxJ^{-1}, [D, y]]$ are compact operators. Here, ϵ, ϵ' are ± 1, obeying the sign-convention described, e.g. in "An Introduction to Noncommutative Geometry", by J. C. Varilly (European Math. Soc., 2006).
- For the even case, additional requirement is some commutation relation of the form $J\gamma = \epsilon'' \gamma J$ for some $\epsilon'' = \pm 1$ between J and the grading operator γ.

Let $\mathbf{Q}'_{\rm real}$ be the subcategory of $\mathbf{Q}'$ of orientation preserving quantum isometries consisting of those $(\mathcal{Q}, U)$ for which following holds on $\mathcal{D}_0$ (the linear span of eigenvectors of D): $(\tilde{J} \otimes \tilde{J}_{\mathcal{Q}}) \circ U = U \circ \tilde{J}$., where $\tilde{J}_{\mathcal{Q}}(x) = x^*$, for x in $\mathcal{Q}_0$ (canonical Hopf algebra of $\mathcal{Q}$.

Theorem

The category $\mathbf{Q}'_{\rm real}$ admits a universal object, denoted by $\widetilde{QISO}_{\rm real}(D)$

Denote by $QISO_{\rm real}(D)$ the C^* algebra generated by elements of the form $\{< (\xi \otimes 1), {\rm ad}_{U_0}(a)(\eta \otimes 1) >, \ a \in \mathcal{A}^\infty\}$, where U_0 is the unitary representation of $\widetilde{QISO}_{\rm real}(D)$ on $\mathcal{H}$. As an important example, Dabrowski et al computed $\widetilde{QISO}_{\rm real}$ for the finite part of the Connes-Chamseddine spectral triple for standard model of NCG.

Quantum isometry for metric spaces

In the next two slides, let us fix a compact metric space (X, d) (without any extra geometric structure).

- For any faithful C^* action β of a CQG $\mathcal{S}$ on $C(X)$, the antipode, say κ, of $\mathcal{S}$ is bounded, so $(\mathrm{id} \otimes \kappa) \circ \beta$ is a well-defined and norm-bounded map on $C(X)$.
- Define β to be 'isometric' in the metric space sense if $(\mathrm{id}_{\mathcal{C}} \otimes \beta)(d) = \sigma_{23} \circ ((\mathrm{id}_{\mathcal{C}} \otimes \kappa) \circ \beta \otimes \mathrm{id}_{\mathcal{C}})(d)$, where σ_{23} denotes the flip of the second and third tensor copies.
- For $\mathcal{S} = C(G)$ for a group G, this definition indeed coincides with the usual definition of isometry.

Theorem

If (X, d) is isometrically embeddable in some $\mathbb{R}^n$ (with Euclidean metric) then there exists a universal CQG $QISO^{\mathrm{metric}}(X, d)$ in the category of CQG's acting isometrically on X.

Actually, the existence theorem extends to a bigger class:

Corollary

Let (X, d) be a compact metric space. Suppose also that there are topological embedding $f : X \to \mathbb{R}^n$ and a homeomorphism ψ of $\mathbb{R}^+$ such that $(\psi \circ d)(x, y) = d_0(f(x), f(y))$ for all $x, y \in X$, where we have denoted the Euclidean metric of $\mathbb{R}^n$ by d_0. Then the conclusion of the above theorem holds.

It is known that an arbitrary finite metric space satisfies the condition of the above corollary with $\psi(t) = t^c$ for some $c > 0$. Thus, our existence theorem does extend that of Banica for finite spaces. Examples of metric spaces satisfying the condition of the corollary also include the spheres S^n (geodesic distance) for all $n \geq 1$.

Only classical symmetries for classical connected spaces

Natural question: what are quantum isometries of a classical manifold?

- Any disconnected compact space with at least 4 components admits a natural faithful action by quantum permutation group of 4 objects, which is a genuine CQG.
- Explicit computations for spheres, tori, G/K for certain homogeneous spaces associated with compact connected semisimple G etc gave QISO=ISO
- Banica, de Commer and Bhowmick showed that many known genuine CQG's cannot act faithfully isometrically on a connected compact manifold.
- These led to the question: can there be a faithful action of a genuine CQG on a compact connected space?
- H. Huang came up with such compact connected metric spaces X, but they were not smooth manifolds.
- Finally, Das, Goswami, Joardar proved the following:

Theorem

If a CQG $\mathcal{Q}$ has a faifthful isometric action on $C(M)$ where M is compact connected Riemannian manifold, then $\mathcal{Q} \cong C(G)$ for some subgroup G of the group of Riemannian isometries of M. In particular, the quantum isometry group of M is $C(Iso(M))$.

This leads us to conjecture that there cannot be a genuine CQG acting smoothly (defined below) and faithfully on a compact, connected, smooth manifold.

Definition

An action α of a CQG $\mathcal{Q}$ on $C(M)$ (where M is a smooth compact manifold) will be called smooth if it maps $C^\infty(M)$ to $C^\infty(M, \mathcal{Q})$ and the span of $\alpha(C^\infty(M))(1 \otimes \mathcal{Q})$ is dense in $C^\infty(M, \mathcal{Q})$ in the natural Frechet topology.

Physical implications of the conjecture

If true, it would mean the following in physical terms:

- For a classical mechanical system with phase-space modeled on a compact connected manifold, the generalized notion of symmetries in terms of quantum groups coincides with the conventional notion, i.e. symmetries coming from group actions only.
- All (quantum) symmetries of a physical model obtained by suitable deformation of a classical model with connected compact phase space, are indeed deformations of the classical (group) symmetries of the original classical model.

No quanutm symmetry for smooth actions?

- There was an announcement of a proof of the smooth no-go conjecture, but it contained a gap, leading to only the no-go result for isometric actions so far.
- However, We proved the conjecture at least for *finite dimensional* CQG.
- The conjecture will follow if we can prove the following: any smooth CQG action on M is isomertric for some choice of Riemannian metric. The wrong announcement had a gap in the proof of this fact using an averaging.

Algebraic no-go results

- Etingof and Walton (2014) proved that, if a finite dimensional, semisimple Hopf algebra inner faithfully (same as what we call faithful) co-acts on a commutative domain then the Hopf algebra must be commutative.
- However, the proof of the above result depends crucially on finite dimension and semisimplicity, so cannot be generalized further.
- The speaker and Etingof, Walton, Mandal proved recently the following: given a commutative unital algebra $\mathcal{A}$ with a finite dimensional generating subspace V which is 'quadratically independent', i.e. the natural map from the symmetric tensor product $S^2(V)$ to $\mathcal{A}$ is one-to-one and a faithful co-action of a CQG Hopf algebra $\mathcal{Q}$ on $\mathcal{A}$ such that V is left invariant, $\mathcal{Q}$ must be commutative, i.e. classical.

- A recent result (not yet written up): given a smooth, real or complex algebraic variety of co-dimension one (i.e. hypersurface), there cannot be any genuine (non-classical) CQG Hopf algebra which can have a linear and faithful co-action on the corresponding coordinate algebra.
- Without smoothness, there can be faithful co-action of even finite dimensional CQG, e.g. $C^*(S_3)$ (group algebra of the permutation group of three elements) on the variety $\{(x,y) : xy = 0\}$, given by Etingof and Walton.
- On the other hand, there can be faithful, linear co-action on $k[x_1, \ldots, x_n]$ of genuine (non-commutative as algebra) Hopf algebras which are of noncompact type (Walton et al).
- **Conjecture** : there cannot be any genuine CQG Hopf algebra co-acting faithfully on the coordinate algebra of any smooth real or complex algebraic variety.

Sketch of proof of no-go for QISO

Lemma

Key Lemma

Let $W \subset \mathbb{R}^N$ have nonempty interior, α a faithful action of a CQG $\mathcal{Q}$ on $C(W)$ which is affine, i.e. α leaves invariant $\mathrm{Sp}\{1, X_1, \ldots, X_N\}$, X_i's being the coordinate functions on W. Then $\mathcal{Q} \cong C(G)$ for some group G.

A lift or (co)-representation of a smooth action α on $\Omega^1(M)$ is a continuous, co-associative $\mathbb{C}$-linear map Γ from $\Omega^1(M)$ to $\Omega^1(M, \mathcal{Q})$ s.t. $\Gamma(df) = (d \otimes \mathrm{id})(\alpha(f))$, $\Gamma(\xi f) = \Gamma(\xi)\alpha(f) = \alpha(f)\Gamma(\xi) \; \forall \; \xi \in \Omega^1(M), f \in C^\infty(M)$. Unlike group actions, Hopf algebra co-action may not have such a lift: e.g. the co-action given by $\alpha(x) = x \otimes a + 1 \otimes b$ on $\mathbb{R}[x]$ of the quantum $ax + b$ Hopf algebra $< a, a^{-1}, b | \; aa^{-1} = a^{-1}a = 1, ab = q^2 ba >$ (coproduct $\Delta(a) = a \otimes a, \Delta(b) = a \otimes b + b \otimes 1$,) is not liftable.

Theorem

Given a smooth action α of $\mathcal{Q}$ on a compact manifold M the following are equivalent:
(i) α admits a lift.
(ii) α is isometric w.r.t. some Riemannian structure on M.
(iii) $\forall x \in M$, the algebra $\mathcal{Q}_x$ generated by $\{(\chi \otimes \mathrm{id})(\alpha(f)), \alpha(g),\ f, g \in C^\infty(M), \chi \in \chi(M)\}$ is commutative, where $\chi(M)$ is the set of all smooth vector fields.

- starting with an isometric action α on M, we can further have $\Gamma^{(k)} := d\alpha^{(k)}$ on $\Lambda^k(M)$ (module of k-foms) which is a co-representation and equivariant, i.e. $<< \Gamma(\omega), \Gamma(\eta) >>_{C^\infty(M,\mathcal{Q})} = \alpha(<< \omega, \eta >>_{C^\infty(M)})$.
- We can view Γ as a co-representation on $\chi(M)$ as well, using the identification of $\chi(M)$ and $\Omega^1(M)$ coming from the Riemannian inner product.

- Using isometry, we show that the Levi-civita connection is 'preserved' in the following sense: $\Gamma(\nabla_X(Y)) = \tilde{\nabla}_{\Gamma(X)}(\Gamma(Y))$. Here, ∇ is the covariant derivative operator corresponding to the Levi-civita connection and $\tilde{\nabla}$ is the extension of ∇ on (topological tensor product) $\chi(M) \otimes \mathcal{Q}$ satisfying $\tilde{\nabla}_{X_1 \otimes q_1}(X_2 \otimes q_2) := \nabla_{X_1}(X_2) \otimes q_1 q_2$.
- This implies $\alpha(\phi \Gamma^k_{ij}) \in \mathcal{Q}_x$, where ϕ is any smooth function supported in the domain of some coordinate chart for which the Christoffel symbols are denoted by Γ^k_{ij}.
- With little more calculations, this further implies a second order commutativity: for each $m \in M$ and local coordinates $(x_1, \ldots, x_n)$ around m, the algebra generated by $\{\alpha(f)(m), \frac{\partial}{\partial x_i}(\alpha(g))(m), \frac{\partial^2}{\partial x_i \partial x_j}(\alpha(h))(m) : f, g, h \in C^\infty(M)\}$ is commutative.

Step 3: Lifting it further

- Commutativity of $\mathcal{Q}_x$ further allows us to prove that Γ naturally induces a $*$-homomorphic action $\tilde{\alpha}$ (say) on $C^\infty(T(O_M))$, where $T(O_M)$ is the total space of the orthonormal frame bundle on M, identifying $C^\infty(T(O_M))$ with suitable completion of the symmetric algebra of the $C^\infty(M)$-module $\Omega^1(M)$.
- The second order commutativity of α implies first order commutativity for the lift $\tilde{\alpha}$, hence we get a Riemannian structure on E for which $\tilde{\alpha}$ is isometric.
- As E is parallelizable, hence has an embedding in some $\mathbb{R}^m$ with trivial normal bundle (w.r.t. the Riemannian metric chosen above) say $N(E)$, lift $\tilde{\alpha}$ as an isometric action Φ on some suitable ϵ-neighbourhood W of E in the total space $N \equiv \mathbb{R}^m$ of $N(E)$.
- Finally, we prove Φ to be affine, then by the key lemma the proof is complete.

Step 4: Proof of affine-ness of Φ

D.Goswami

- Let $D_i^k = \frac{\partial}{\partial y_i}\Phi(y_k)$, $D_{ij}^k = \frac{\partial^2}{\partial y_i \partial y_j}\Phi(y_k)$, where $y_1, \ldots, y_m$ are the standard coordinates of $\mathbb{R}^m$. As $Int(W)$ is open connected, it suffices to show that $D_{ij}^k = 0$ for all k, i, j.
- As isometric actions satisfy second order commutativity, D_{ij}^k and D_m^l commute.
- By the isometry condition of Φ:

$$\sum_{l=1}^{N} D_i^l D_j^l = \delta_{ij} 1. \tag{1}$$

- Applying $\frac{\partial}{\partial y_k}$ to equation (1), and using the commutativity of D^l_{jk} and D^l_i's

$$\sum_{l=1}^{N}(D^l_{ik}D^l_j + D^l_{jk}D^l_i) = 0. \tag{2}$$

- $A_{n^2\times n} \equiv ((A_{(ij),k}))$, with $A_{(ij),k} = D^k_{ij}$, $B_{n\times n} = ((B_{ij} = D^i_j))$, $C := AB$.
- From (2)

$$C_{(ik)j} + C_{(jk)i} = 0. \tag{3}$$

- As $C_{(ij)k} = C_{(ji)k}$ for all i, j, k, equation (3) gives

$$C_{(ik)j} = C_{(ki)j} = -C_{(ji)k} = -C_{(ij)k} = C_{(kj)i} = C_{(jk)i}.$$

- So again by equation (3), $C_{(ik)j} = 0$ for all i, j, k i.e. $C = 0$, hence $A = 0$ as B is unitary.

Noncommutative Tori (Bhowmick-Goswami 2009)

Consider the noncommutative two-torus $\mathcal{A}_\theta$ (θ irrational) generated by two unitaries U, V satisfying $UV = e^{2\pi i\theta}VU$, and the standard spectral triple on it described by Connes. Here, $\mathcal{A}^\infty$ is the unital $*$-algebra spanned by U, V; $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ (where τ is the unique faithful trace on $\mathcal{A}_\theta$) and D is given by $D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$, where d_1 and d_2 are closed unbounded linear maps on $L^2(\tau)$ given by $d_1(U^mV^n) = mU^mV^n$, $d_2(U^mV^n) = nU^mV^n$. 'Laplacian' $\mathcal{L}$ given by $\mathcal{L}(U^mV^n) = -(m^2+n^2)U^mV^n$.

Theorem

(i) $QISO^{\mathcal{L}} = \oplus_{k=1}^{8} C^*(U_{k1}, U_{k2})$ *(as a* C^* *algebra), where for odd* k, U_{k1}, U_{k2} *are the two commuting unitary generators of* $C(\mathbb{T}^2)$, *and for even* k, $U_{k1}U_{k2} = \exp(4\pi i\theta)U_{k2}U_{k1}$,
(ii) $QISO^+(D) \cong C(\mathbb{T}^2)$.

$SU_\mu(2)$ (Bhowmick-Goswami 2009)

- The CQG $SU_\mu(2)$ $\mu \in [-1, 1]$ is the universal unital C^* algebra generated by α, γ satisfying: $\alpha^*\alpha + \gamma^*\gamma = 1$, $\alpha\alpha^* + \mu^2\gamma\gamma^* = 1$, $\gamma\gamma^* = \gamma^*\gamma$, $\mu\gamma\alpha = \alpha\gamma$, $\mu\gamma^*\alpha = \alpha\gamma^*$., and the coproduct given by : $\Delta(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma$, $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.
- On the Hilbert space $L^2(h)$ (h Haar state), Chakraborty-Pal described a natural spectral triple with the D given by $D(e_{ij}^{(n)}) = (2n+1)e_{ij}^{(n)}$ if $n \neq i$, and $-(2n+1)e_{ij}^{(n)}$ for $n = i$, where $e_{ij}^{(n)}$ are normalised matrix elements of the $2n+1$ dimensional irreducible representation, n being half-integers.
- $QISO^+(D) \cong U_\mu(2) = C^*\{u_{ij}, i, j = 1, 2, | \ ((u_{ij}))\ \text{unitary},\ u_{11}u_{12} = \mu u_{12}u_{11}, u_{11}u_{21} = \mu u_{21}u_{11}, u_{12}u_{22} = \mu u_{22}u_{12}, u_{21}u_{22} = \mu u_{22}u_{21}, u_{12}u_{21} = u_{21}u_{12}, u_{11}u_{22} - u_{22}u_{11} = (\mu - \mu^{-1})u_{12}u_{21}\}$.

Podles spheres (Bhowmick-Goswami 2010)

- The Podles sphere $S^2_{\mu,c}$ is the universal C^* algebra generated by A, B satisfying $AB = \mu^{-2}BA, A = A^* = B^*B + A^2 - cI = \mu^{-2}BB^* + \mu^2A^2 - c\mu^{-2}I$.
- $S^2_{\mu,c}$ can also be identified as a suitable C^* subalgebra of $SU_\mu(2)$ and leaves invariant the subspace $\mathcal{K} = \overline{\mathrm{Span}\{e^{(l)}_{\pm\frac{1}{2},\ m} : l = \frac{1}{2}, \frac{3}{2}, \ldots,\ m = -l, -l+1, \ldots l\}}$ of $L^2(SU_\mu(2), h)$.
- R-twisted spectral triple given by: $D(e^{(l)}_{\pm\frac{1}{2},\ m}) = (c_1 l + c_2)e^{(l)}_{\mp\frac{1}{2},\ m}$, (where $c_1, c_2 \in \mathbb{R},\ c_1 \neq 0$), $R(e^{(n)}_{\pm\frac{1}{2},\ i}) = \mu^{-2i}e^{(n)}_{\pm\frac{1}{2},\ i}$.
- $$QISO^+_R(D) = SO_\mu(3) \equiv C^*\left(e^{(1)}_{ij},\ i,j = -1,0,1\right).$$

Free and half liberated spheres (Banica-Goswami)

- Free sphere: $A_n^+ = C^*\left(x_1, \ldots, x_n \middle| x_i = x_i^*, \sum x_i^2 = 1\right)$.
- It has a faithful trace, and in the corresponding GNS space we can construct a spectral triple for which the quantum isometry group is the free orthogonal group

$$O_n^+ = C^*\left(u_{11}, \ldots, u_{nn} \middle| u_{ij} = u_{ij}^*, u^t = u^{-1}\right).$$

- Similarly, consider the half-liberated sphere: $A_n^* = C^*\left(x_1, \ldots, x_n \middle| x_i = x_i^*, x_i x_j x_k = x_k x_j x_i, \sum x_i^2 = 1\right)$.
- Again, for a natural spectral triple on this, we get the following the quantum isometry group: $O_n^* = C^*\left(u_{11}, \ldots, u_{nn} \middle| u_{ij} = u_{ij}^*, u_{ij}u_{kl}u_{st} = u_{st}u_{kl}u_{ij}, u^t = u^{-1}\right)$.

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Sketch of proof for existence of $QISO^{\mathcal{L}}$

Let us give some ideas of a typical construction of quantum isometry groups. Consider the approach based on Laplacian.

- Let $\{e_{ij}, j = 1, \ldots, d_i; i = 1, 2, \ldots\}$ be the complete list of eigenvectors of the Laplacian $\mathcal{L}$, $\{e_{ij}, j = 1, \ldots, d_i\}$ being the (orthonormal) basis for i-th eigenspace. recall that these are actually elements of $\mathcal{A}^\infty$, and let $\mathcal{A}_0^\infty$ be the span of these elements which is norm-dense in $\mathcal{A}$ by assumption
- We have to use the formalism of isometric quantum family. Call $(\mathcal{S}, \alpha)$ be such a family if $\mathcal{S}$ is a unital C^* algebra and $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{S}$ is a $*$-homomorphism which commutes with $\mathcal{L}$, ie isometric, and also the linear span of $\alpha(\mathcal{A})(1 \otimes \mathcal{S})$ is norm dense in $\mathcal{A} \otimes \mathcal{S}$.
- We first claim that the 'connectedness assumption' that $\ker(\mathcal{L}) = \mathbb{C}1$ implies α preserves the volume form τ.

- Proof of claim: for any state ϕ on $\mathcal{S}$, consider the linear map $C := \alpha_\phi = (\mathrm{id} \otimes \phi) \circ \alpha$ on $\mathcal{A}_0^\infty$ which commutes with the self-adjoint operator $\mathcal{L}$, so leaves invariant each eigenspace, in particular maps the vector 1 to itself, and its orthocomplement (which is the direct sum of eigenspaces of $\mathcal{L}$) to itself too. For $a \in \mathcal{A}_0^\infty$, $< 1, (a - \tau(a)1) >= 0$, so $\tau(C(a)) - \tau(a) =< 1, C(a - \tau(a)1) >= 0$.
- Thus, α extends to a unitary operator from $\mathcal{H} \otimes \mathcal{S} = L^2(\mathcal{A}, \tau) \otimes \mathcal{S}$ to , which maps $e_{ij} \otimes 1$ to say $\sum_k e_{ik} \otimes q_{kj}^{(i)}$, and tracial property of τ implies that $(q_{kj}^{(i)})$ give a copy of $A_u(I_{d_i})$. This identifies $\mathcal{S}$ as a quotient of $*_i A_u(I_{d_i})$, say w.r.t. the ideal $\mathcal{I}_{\mathcal{S}}$.
- Now consider all the ideals of the form $\mathcal{I}_{\mathcal{S}}$ as above and take their intersection, say $\mathcal{I}$. One can prove that $(*_i A_u(I_{d_i})) / \mathcal{I}$ is the universal quantum family of isometries and is also a CQG, which is indeed the desired $QISO^{\mathcal{L}}$.

Open problems to be investigated

- Proving some general results about the structure and representation theory of such quantum isometry groups.
- Extending the formulation of quantum isometry groups to the set-up of possibly noncompact manifolds (both classical and noncommutative), where one has to work in the category of locally compact quantum groups.
- Formulating a definition (and proving existence) of a quantum group of isometry for compact metric spaces , and more generally, for quantum metric spaces in the sense of Rieffel. Some wrok in this direction is done by Sabbe and Quaegebeur recently.

QISO of deformed noncommutative manifolds (Bhowmick-Goswami 2009)

Recall Rieffel deformation of C^* algebras and Rieffel-Wang deformation of CQG. we give a general scheme for computing quantum isometry groups by proving that $\widetilde{QISO_R^+}$ of a deformed noncommutative manifold coincides with (under reasonable assumptions) a similar (Rieffel-Wang) deformation of the $\widetilde{QISO_R^+}$ of the original manifold.

Let $(\mathcal{A}, \mathbb{T}^n, \beta)$ be a C^* dynamical system, $\mathcal{A}^\infty$ be the algebra of smooth (C^∞) elements for the action β., and D be a self-adjoint operator on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is an R-twisted, θ-summable spectral triple of compact type. Assume that there exists a compact abelian group $\widetilde{\mathbb{T}^n}$ with a covering map $\gamma : \widetilde{\mathbb{T}^n} \to \mathbb{T}^n$, and a strongly continuous unitary representation $V_{\tilde{g}}$ of $\widetilde{\mathbb{T}^n}$ on $\mathcal{H}$ such that

$$V_{\tilde{g}} D = D V_{\tilde{g}}, \ \ V_{\tilde{g}} a V_{\tilde{g}}^{-1} = \beta_g(a), \ \ g = \gamma(\tilde{g}).$$

Theorem

(i) For each skew symmetric $n \times n$ real matrix J, there is a natural representation of the Rieffel-deformed C^* algebra $\mathcal{A}_J$ in $\mathcal{H}$, and $(\mathcal{A}_J^\infty = (\mathcal{A}^\infty)_J, \mathcal{H}, D)$ is an R-twisted spctral triple of compact type.
(ii) If $QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D)$ and $(QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}$ have C^* actions on $\mathcal{A}$ and $\mathcal{A}_J$ respectively, where $\tilde{J} = J \oplus (-J)$, we have

$$QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D) \cong (QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}.$$

(iii) A similar conclusion holds for $QISO^+(\mathcal{A}^\infty)$, $QISO^+(\mathcal{A}_J^\infty)$ provided they exist.
(iv) In particular, for deformations of classical spectral triples, the C^* action hypothesis of (ii) or (iii) hold, and hence the above conclusions hold too.

Fix a unital commutative k-algebra $\mathcal{A}$ with a finite-dimensional generating subspace V that is quadratically independent. Let $\mathcal{Q}$ be a Hopf algebra that coacts on $\mathcal{A}$ inner-faithfully leaving V invariant.

Lemma

Suppose that V is an inner-faithful finite-dimensional comodule over a Hopf algebra $\mathcal{Q}$, and assume that the decomposition $V \otimes V = S^2 V \oplus \wedge^2 V$ is preserved by this coaction. Here, $\wedge^2 V := (V \otimes V)/(v \otimes w + w \otimes v)_{v,w \in V}$. Then, $\mathcal{Q}$ is commutative.

Proof:
Let the coaction α be given by $\alpha(v) = T(v \otimes 1)$, for $T \in End(V) \otimes \mathcal{Q}$. Consider the natural $\mathcal{Q}$-coaction on $V \otimes V$ defined by the matrix $T^{13}T^{23} \in End(V \otimes V) \otimes \mathcal{Q}$. Here, $T^{13} = \sum_{i,j} E_{ij} \otimes Id \otimes t_{ij}$, and $T^{23} = \sum_{i,j} Id \otimes E_{ij} \otimes t_{ij}$, for the elementary matrices E_{ij}. The hypotheses imply that $T^{13}T^{23}$ lies in $(End(S^2V) \otimes \mathcal{Q}) \oplus (End(\wedge^2 V) \otimes \mathcal{Q})$, hence it commutes with the permutation that flips the two copies of V. Thus, $T^{13}T^{23} = T^{23}T^{13}$, so matrix elements of T commute with each other. Since matrix elements of T generate $\mathcal{Q}$ (by the inner-faithfulness of V), we obtain that $\mathcal{Q}$ is commutative.

Theorem

If the co-action of $\mathcal{Q}$ preserves a nondegenerate bilinear form B on V, then $\mathcal{Q}$ is commutative.

Proof:
The form B defines an invariant nondegenerate form B^2 on $V \otimes V$ given by $B^2(a \otimes b, c \otimes d) = B(a, d)B(b, c)$. Now the co-action of $\mathcal{Q}$ on $\mathcal{A}$ is induced from the natural coaction of $\mathcal{Q}$ on $V \otimes V$, and by hypothesis of quadratic independence, $\wedge^2 V$ is $\mathcal{Q}$-invariant. Thus, the orthogonal complement to $\wedge^2 V$ in $V \otimes V$ under the form B^2, i.e. $S^2 V$ is also invariant under the coaction of $\mathcal{Q}$ on $V \otimes V$. The previous lemma now proves the theorem.

Adapting this proof to the case of a hermitian inner product we can get a similar result for Hopf $*$-algebra preserving an inner product. Moreover, as every finite dimensional co-module of a CQG Hopf algebra can be made into a unitary co-representation by choosing a suitable inner product, we conclude that

Theorem

if a CQG Hopf algebra co-acts on $\mathcal{A}$, inner faithfully and leaving V invariant, then $\mathcal{Q}$ must be commutative.