# Unbounded Kasparov products by differentiable Hilbert $C^*$-modules

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# $KK$-theory

## Setting

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2. *Kasparov's bivariant $K$-theory denoted by $KK_*(A, B)$.*

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**Remark**

*We have*

$$KK_*(\mathbb{C}, B) \cong K_*(B) \qquad \text{and} \qquad KK_*(A, \mathbb{C}) \cong K^*(A)$$

*where $K_*(B)$ is the $K$-theory of $B$ and $K^*(A)$ is the $K$-homology of $A$.*

# Interior Kasparov product

**Theorem (Kasparov)**

*There exists a bilinear and associative pairing*

$$\widehat{\otimes}_B : KK(A, B) \times KK(B, C) \to KK(A, C)$$

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**Remark**

*The interior Kasparov product*

$$\widehat{\otimes}_B : KK(\mathbb{C}, B) \times KK(B, \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$$

*recovers the index pairing.*

# Norm-structure on $*$-algebras

## Convention

1. *All $*$-algebras $\mathscr{A}$, $\mathscr{B}$ in this talk come equipped with a $C^*$-norm $\|\cdot\|$ and an operator space norm $\|\cdot\|_1$. The $C^*$-closures are denoted by $A$ and $B$ and the operator space closures are denoted by $A_1$, $B_1$.*

# Norm-structure on $*$-algebras

**Convention**

1. *All $*$-algebras $\mathscr{A}$, $\mathscr{B}$ in this talk come equipped with a $C^*$-norm $\| \cdot \|$ and an operator space norm $\| \cdot \|_1$.*
   *The $C^*$-closures are denoted by $A$ and $B$ and the operator space closures are denoted by $A_1$, $B_1$.*
2. *The inclusion $\mathscr{A} \to A$ is supposed to induce a completely bounded linear map $A_1 \to A$ (and similarly for $\mathscr{B} \to B$).*
   *Thus, there exists a constant $C > 0$ such that*

$$\|x\| \leq C \cdot \|x\|_1 \qquad \forall x \in M_n(\mathscr{A}) \, , \; n \in \mathbb{N}$$

# Unbounded modular cycles (1)

**Definition**

*An* **unbounded modular cycle** *from* $\mathscr{B}$ *to* $C$ *consists of*

1. *A countably generated* $C^*$*-correspondence* $Y$ *from* $B$ *to* $C$

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3. *A bounded positive operator* $\Gamma : Y \to Y$ *with dense image*

*such that the following holds:*

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2. $D\pi(b)\Gamma - \Gamma\pi(b)D : \mathscr{D}(D) \to Y$ *extends to a bounded operator* $d_\Gamma(b) : X \to X$ *for all* $b \in \widetilde{\mathscr{B}}$

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4. *The linear map* $\rho_\Gamma : \mathscr{B} \to \mathscr{L}(Y)$ *is completely bounded.*
5. *The sequence* $\{\pi(b)\Gamma(\Gamma + 1/n)^{-1}\}$ *converges in operator norm to* $\pi(b)$ *for all* $b \in B$

# The bounded transform

**Theorem (K.)**

*Suppose that $(Y, D, \Gamma)$ is an unbounded modular cycle from $\mathscr{B}$ to $C$. Then the pair $(Y, D(1 + D^2)^{-1/2})$ is a bounded Kasparov module from $B$ to $C$.*
*The associated class in KK-theory is denoted by $[D] \in KK_*(B, C)$.*

# The modular transform

## Definition

*The* **modular transform** *of* $(Y, D, \Gamma)$ *is the unbounded operator*

$$G_{D,\Gamma} : \Gamma\big(\mathscr{D}(D)\big) \to Y$$

*defined by*

$$G_{D,\Gamma} : \Gamma(\xi) \mapsto \frac{1}{\pi} \int_0^\infty \lambda^{-1/2}(1 + \lambda\Gamma^2 + D^2)^{-1}\Gamma D\big(\Gamma(\xi)\big)\, d\lambda$$

*for all* $\xi \in \mathscr{D}(D)$.

# Differentiable modules

**Definition**

*A $C^*$-correspondence $X$ from $A$ to $B$ is* **differentiable** *(w.r.t. $\mathscr{A}$ and $\mathscr{B}$) when there exists a generating sequence $\{\xi_n\}$ for $X$ such that*

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1. *The sequence $\Big\{ \sum_{n,m=1}^N \langle \xi_n, \pi(a)\xi_m \rangle \delta_{n,m} \Big\}_{N=1}^\infty$ is Cauchy in $M(\mathscr{B})$ for all $a \in \widetilde{\mathscr{A}}$*

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2. *The linear map $\tau : \mathscr{A} \to \mathscr{K}(B_1)$ defined by*

$$\tau(a) := \sum_{n,m=1}^\infty \langle \xi_n, \pi(a)\xi_m \rangle \delta_{n,m}$$

*is completely bounded.*

# Main question

## Setting

1. *A differentiable $C^*$-correspondence $X$ from $\mathscr{A}$ to $\mathscr{B}$ such that $\pi_A(a) \in \mathscr{K}(X)$ for all $a \in A$*

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1. *A differentiable $C^*$-correspondence $X$ from $\mathscr{A}$ to $\mathscr{B}$ such that $\pi_A(a) \in \mathscr{K}(X)$ for all $a \in A$*
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2. *An unbounded modular cycle* $(Y, D, \Gamma)$ *from* $\mathscr{B}$ *to* $C$
3. *KK-classes:* $[X] \in KK_0(A, B)$ *and* $[D] \in KK_*(B, C)$.

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3. *$KK$-classes: $[X] \in KK_0(A, B)$ and $[D] \in KK_*(B, C)$.*

## Question

*Can I find an* **explicit** *unbounded modular cycle*

$$(X \widehat{\otimes}_B Y, D_\Delta, \Delta)$$

*from $\mathscr{A}$ to $C$ such that $[D_\Delta] = [X]\widehat{\otimes}_B[D]$ in $KK_*(A, C)$?*

# The modular lift

## Definition

*Fix a differentiable generating sequence $\{\xi_n\}$ for $X$.*
*Define the bounded adjointable operator*

$$\Phi : X \widehat{\otimes}_B Y \to \ell^2(Y) \qquad x \otimes_B y \mapsto \sum_{n=1}^{\infty} \pi_B\big(\langle \xi_n, x \rangle\big)(y)\, \delta_n$$

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**Definition**

*The* **modular lift** *is the closure of*

$$\Phi^*(1 \otimes D)\Phi : \mathscr{D}\big((1 \otimes D)\Phi\big) \to X \widehat{\otimes}_B Y$$

*The modular lift is denoted by* $D_\Delta$.

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**Theorem (K.)**

*The triple* $\big(X \widehat{\otimes}_B Y, D_\Delta, \Delta\big)$ *is an unbounded modular cycle from* $\mathscr{A}$ *to* $C$.

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# The unbounded Kasparov product (2)

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*We have the identity*

$$[D_\Delta] = [X] \widehat{\otimes}_B [D]$$

*where* $\widehat{\otimes}_B : KK_0(A, B) \times KK_*(B, C) \to KK_*(A, C)$ *denotes the interior Kasparov product.*

# Example: Fractal string (1)

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1. *Define* $\mathscr{A} \subseteq \mathscr{B}$ as *smallest* $*$*-subalgebra of* $\mathscr{B}$ *such that* $x_n b x_n^* \in \mathscr{A}$ *for all* $b \in \mathscr{B}$, $n \in \mathbb{N}$.

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2. *Define* $C^*$*-correspondence* $X$ *from* $A$ *to* $B$ *as* $C^*$*-closure of* $\text{span}\{x_n b \mid b \in B, n \in \mathbb{N}\} \subseteq B$.

# Example: Fractal string (2)

## Definition

1. *Define Hilbert space* $H \subseteq G$ *as closure of* $\text{span}\{\pi(x_n)(\eta) \mid \eta \in G, n \in \mathbb{N}\} \subseteq G$.

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2. *Define modular lift* $D_\Delta : \mathscr{D}(D_\Delta) \to H$ *as closure of*

$$\sum_{n=1}^{\infty} \pi(\xi_n) D \pi(\xi_n^*) : \text{span}\{\pi(x_n)(\eta) \mid \eta \in \mathscr{D}(D), n \in \mathbb{N}\} \to H$$

*where* $\xi_n := x_n / n$ *for all* $n \in \mathbb{N}$.

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   *where* $\xi_n := x_n / n$ *for all* $n \in \mathbb{N}$.
3. *Define modular operator* $\Delta := \sum_{n=1}^{\infty} \pi(\xi_n \xi_n^*) : H \to H$.

# Fractal strings (3)

**Corollary**

*The triple* $(H, D_\Delta, \Delta)$ *is an unbounded modular cycle from* $\mathscr{A}$ *to* $\mathbb{C}$ *and the identity* $[D_\Delta] = [X]\widehat{\otimes}_B[D]$ *holds in the* $K$*-homology group* $K^*(A)$*.*