SOME NATURALLY DEFINED STAR PRODUCTS FOR KÄHLER MANIFOLDS

Martin Schlichenmaier

Mathematics Research Unit
University of Luxembourg

SISSA workshop, Hilbert modules and index theory, 29.9.-2.10.2015
Travaux mathématiques, Vol. 20 (2012), 187-204

- One mathematical aspect of quantization is the passage from the commutative world to the non-commutative world.
- one way a deformation quantization (also called star product)
- can only be done on the level of formal power series over the algebra of functions
- was pinned down in a mathematically satisfactory manner by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer.

OUTLINE

- give an overview of some naturally defined star products in the case that our “phase-space manifold” is a (compact) Kähler manifold
- here we have additional complex structure and search for star products respecting it
- yield star products of separation of variables type (Karabegov) resp. Wick or anti-Wick type (Bordemann and Waldmann)
- both constructions are quite different, but there is a 1:1 correspondence (Neumaier)
- still quite a lot of them

- single out certain naturally given ones.
- restrict to quantizable Kähler manifolds
- Berezin-Toeplitz star product, Berezin transform, Berezin star product
- a side result: star product of geometric quantization
- all of the above are equivalent star product, but not the same
- give Deligne-Fedosov class and Karabegov forms
- give the equivalence transformations

GEOMETRIC SET-UP

- (M, ω) a pseudo-Kähler manifold.
  M a complex manifold, and ω, a non-degenerate closed $(1,1)$-form
- if ω is a positive form then (M, ω) is a honest Kähler manifold
- $C^\infty(M)$ the algebra of complex-valued differentiable functions with associative product given by point-wise multiplication
- define the Poisson bracket

$$\{ f, g \} := \omega(X_f, X_g) \qquad \omega(X_f, \cdot) = df(\cdot)$$

- $C^\infty(M)$ becomes a Poisson algebra.

Star product

star product for M is an associative product $\star$ on $\mathcal{A} := C^\infty(M)[[\nu]]$, such

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f)/\nu = -\mathrm{i}\{f, g\} \mod \nu$.

Also

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M),$$

differential (or local) if $C_k(\ ,\ )$ are bidifferential operators.
Usually: $1 \star f = f \star 1 = f$.

Equivalence of star products

$\star$ and $\star'$ (the same Poisson structure) are *equivalent* means there exists
a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ and $B(f) \star' B(g) = B(f \star g)$.

to every equivalence class of a differential star product one assigns its Deligne-Fedosov class

$$cl(\star) \in \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M, \mathbb{C})[[\nu]]).$$

gives a 1:1 correspondence

Existence: by DeWilde-Lecomte, Omori-Maeda-Yoshioka, Fedosov, ..., Kontsevich.

Separation of Variables type

- pseudo-Kähler case: we look for star products adapted to the complex structure
- separation of variables type (Karabegov)
- Wick and anti-Wick type (Bordemann - Waldmann)
- Karabegov convention: of separation of variables type if in $C_k(.,.)$ for $k \geq 1$ the first argument differentiated in anti-holomorphic and the second argument in holomorphic directions.
- we call this convention separation of variables (anti-Wick) type and call a star product of separation of variables (Wick) type if the role of the variables is switched
- we need both conventions

Karabegov construction (sketch of a sketch)

- (M, ω_{-1}) the pseudo-Kähler manifold
- a formal deformation of the form $(1/\nu)\omega_{-1}$ is a formal form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\,\omega_1 + \ldots$$

$\omega_r,\ r \geq 0$, closed (1,1)-forms on M.

- Karabegov: to every such $\widehat{\omega}$ there exists a star product $\star$ of anti-Wick type
- and vice-versa
- Karabegov form of the star product $\star$ is $kf(\star) := \widehat{\omega}$,
- the star product $\star_K$ with classifying Karabegov form $(1/\nu)\omega_{-1}$ is Karabegov's standard star product.

- Formal Berezin transform
- for local antiholomorphic functions a and holomorphic functions b on $U \subset M$ we have the relation

$$a \star b = I_\star(b \star a) = I_\star(b \cdot a),$$

- can be written as

$$I_\star = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \quad I_0 = id, \quad I_1 = \Delta.$$

- the formal Berezin transform $I_\star$ determines the $\star$ uniquely.

- Start with $\star$ separation of variables type (anti-Wick) (M, ω_{-1})
- opposite of the dual

$$f \star' g = I^{-1}(I(f) \star I(g)).$$

on (M, ω_{-1}), is of Wick type

- the formal Berezin transform $I_\star$ establishes an equivalence of the star products

$$(\mathcal{A}, \star) \text{ and } (\mathcal{A}, \star')$$

Classifying forms

$\star$ star product of anti-Wick type with Karabegov form $kf(\star) = \widehat{\omega}$
Deligne-Fedosov class calculates as

$$cl(\star) = \frac{1}{\mathrm{i}}([\widehat{\omega}] - \frac{\delta}{2}).$$

[..] denotes the de-Rham class of the forms and δ is the canonical class of the manifold i.e. $\delta := c_1(K_M)$.

standard star product $\star_K$ (with Karabegov form $\widehat{\omega} = (1/\nu)\omega_{-1}$)

$$cl(\star_K) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega_{-1}] - \frac{\delta}{2}).$$

- For the Karabegov form to be in 1:1 correspondence, we need to fix a convention: Wick or anti-Wick for reference
- here we refer to the anti-Wick type product
- it $\star$ is of Wick type we set

$$kf(\star) := kf(\star^{op}),$$

- where

$$f \star^{op} g = g \star f$$

is obtained by switching the arguments. It is a star product of (anti-Wick) type for the pseudo-Kähler manifold $(M, -\omega)$

Other general constructions

- Bordemann and Waldmann: modification of Fedosov's geometric existence proof.
- fibre-wise Wick product.
- by a modified Fedosov connection a star product $\star_{BW}$ of Wick type is obtained.
- Karabegov form is $-(1/\nu)\omega$
- Deligne class class

$$cl(\star_{BW}) = -cl(\star_{BW}^{op}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}).$$

Neumaier: by adding a formal closed $(1,1)$ form as parameter each star product of separation of variables type can be obtained by the Bordemann-Waldmann construction

Reshetikhin and Takhtajan:
formal Laplace expansions of formal integrals related to the star product.
coefficients of the star product can be expressed (roughly) by Feynman diagrams

Berezin-Toeplitz Star product

- compact and quantizable Kähler manifold (M, ω),
- quantum line bundle (L, h, ∇), L is a holomorphic line bundle over M, h a hermitian metric on L, ∇ a compatible connection
- recall (M, ω) is quantizable, if there exists such (L, h, ∇), with

$$curv_{(L,\nabla)} = -\mathrm{i}\,\omega$$

- consider all positive tensor powers $(L^m, h^{(m)}, \nabla^{(m)})$, scalar product

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\,\Omega, \qquad \Omega := \frac{1}{n!}\underbrace{\omega \wedge \omega \cdots \wedge \omega}_{n}$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}(M, L^m)$

$$s \quad \mapsto \quad T_f^{(m)}(s) := \Pi^{(m)}(f \cdot s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)$$

the Toeplitz operator of level m.

Berezin-Toeplitz operator quantization

$$f \mapsto \left(T_f^{(m)}\right)_{m\in\mathbb{N}_0}.$$

has the correct semi-classical behavior

Theorem (Bordemann, Meinrenken, and Schl.)

(a)

$$\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$$

(b)

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

(c)

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| = O(1/m)$$

- Statement of the previous theorem corresponds to the fact that we have a continuous field of C^*-algebras (with additionally Dirac condition on commutators).
- over $I = \{0\} \cup \{\frac{1}{m} \in \mathbb{N}\}$,
- over $\{0\}$ we set the algebra $C^\infty(M)$, over $\frac{1}{m}$ the algebra $\mathrm{End}(\Gamma_{hol}(M, L^m))$,
- section is given by $f \in C^\infty(M)$
-

$$f \quad \mapsto \quad (\, f \,,\ T_f^{(m)}, m \in \mathbb{N}).$$

Theorem (BMS, Schl., Karabegov and Schl.)

$\exists$ a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f,g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}$$

Further properties: is of separation of variables type (Wick type)

classifying Deligne-Fedosov class $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ and Karabegov form $\frac{-1}{\nu}\omega + \omega_{can}$

possible: auxiliary hermitian line (or even vector) bundle can be added, meta-plectic correction.

Further result: The Toeplitz map of level m

$$T^{(m)} : C^{\infty}(M) \to End(\Gamma_{hol}(M, L^m))$$

is surjective

implies that the operator $Q_f^{(m)}$ of geometric quantization (with holomorphic polarization) can be written as Toeplitz operator of a function f_m (maybe different for every m)

indeed Tuynman relation:

$$Q_f^{(m)} = \mathrm{i}\ T^{(m)}_{f - \frac{1}{2m}\Delta f}$$

- star product of geometric quantization
- set $B(f) := (id - \nu\frac{\Delta}{2})f$

$$f \star_{GQ} g = B^{-1}(B(f) \star_{BT} B(g))$$

  defines an equivalent star product
- can also be given by the asymptotic expansion of product of geometric quantisation operators
- it is not of separation of variable type
- but equivalent to $\star_{BT}$.

Where is the Berezin star product ??

- It is an important star product: Berezin, Cahen-Gutt-Rawnsley, etc.
- The original definition is limited in applicability.
- We will give a definition for quantizable Kähler manifold.
- Clue: define it as the opposite of the dual of $\star_{BT}$.
- $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- Problem: How to determine I?
- describe the formal I by asymptotic expansion of some geometrically defined $I^{(m)}$

- assume the bundle L is very ample (i.e. has enough global sections)
- pass to its dual $(U, k) := (L^*, h^{-1})$ with dual metric k
- inside of the total space U, consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

- $\tau : Q \to M$ (or $\tau : U \to M$) the projection, coherent vectors/states in the sense of Berezin-Rawnsley-Cahen-Gutt:

$$\langle e_\alpha^{(m)}, s\rangle = \alpha^{\otimes m}(s(\tau(\alpha)))$$

where

$$x \in M \mapsto \alpha = \tau^{-1}(x) \in U \setminus 0 \mapsto e_\alpha^{(m)} \in \Gamma_{hol}(M, L^m)$$

As

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}, \qquad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}\ .$$

we obtain:

$$x \in M \mapsto \mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m))$$

- Bergman projectors $\Pi^{(m)}$, Bergman kernels, ...
- Covariant Berezin symbol $\sigma^{(m)}(A)$ (of level m) of an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$

$$\sigma^{(m)}(A) : M \to \mathbb{C},$$

$$x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, A e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle} = \mathrm{Tr}(A P_x^{(m)})$$

Importance of the covariant symbol

- Construction of the Berezin star product, only for limited classes of manifolds (see Berezin, Cahen-Gutt-Rawnsley)
- $\mathcal{A}^{(m)} \le C^\infty(M)$, of level m covariant symbols.
- symbol map is injective (follows from Toeplitz map surjective)
- for $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed
- $$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$
- $\star_{(m)}$ on $\mathcal{A}^{(m)}$ is an associative and noncommutative product
- Crucial problem, how to obtain from $\star_{(m)}$ a star product for all functions (or symbols) independent from the level m ?

Berezin transform

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

Theorem: (Karabegov - Schl.)
$I^{(m)}(f)$ has a complete asymptotic expansion as $m \to \infty$

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i},$$

$I_i : C^\infty(M) \to C^\infty(M)$, $I_0(f) = f, \qquad I_1(f) = \Delta f.$

- Δ is the Laplacian with respect to the metric given by the Kähler form ω

BEREZIN STAR PRODUCT

- from asymptotic expansion of the Berezin transform get formal expression

$$I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M)$$

- set $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- $\star_B$ is called the Berezin star product
- I gives the equivalence to $\star_{BT}$ ($I_0 = id$). Hence, the same Deligne-Fedosov classes
- if the covariant symbol star product works, it will coincide with the star product $\star_B$.

- separation of variables type (but now of anti-Wick type).
- Karabegov form is $\frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m)$
- u_m is the Bergman kernel $\mathcal{B}_m(\alpha,\beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle$ evaluated along the diagonal
- $\mathbb{F}$ means: take asymptotic expansion in $1/m$ as formal series in ν
- $I = I_{\star_B}$, the geometric Berezin transform equals the formal Berezin transform of Karabegov for $\star_B$
- both star products $\star_B$ and $\star_{BT}$ are dual and opposite to each other

Summary of naturally defined star product

	name	Karabegov form	Deligne Fedosov class
$\star_{BT}$	Berezin-Toeplitz	$\frac{-1}{\nu}\omega + \omega_{can}$ (Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{B}$	Berezin	$\frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m)$ (anti-Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{GQ}$	geometric quantization	(—)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{K}$	standard product	$(1/\nu)\,\omega$ (anti-Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{BW}$	Bordemann-Waldmann	$-(1/\nu)\,\omega$ (Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$.

u_m Bergman kernel evalulated along the diagonal in $Q \times Q$

δ the canonical class of the manifold M

- Berezin transform is not only the equivalence relating $\star_{BT}$ with $\star_B$
- also it (resp. the Karabegov form) can be used to calculate the coefficients of these naturally defined star products,
- either directly
- or with the help of the certain type of graphs (see the very interesting work of Gammelgaard and Hua Xu).

$\tau(\alpha) = x$, $\tau(\beta) = y$ with $\alpha, \beta \in Q$

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta)$$
$$= \frac{1}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \int_M \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle f(y)\Omega(y) \; .$$

Note that:

$$u_m(x) := \mathcal{B}_m(\alpha,\alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle,$$
$$v_m(x,y) := \mathcal{B}_m(\alpha,\beta)\cdot\mathcal{B}_m(\beta,\alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle$$

are well-defined on M and on $M \times M$ respectively.