PULLING BACK ASSOCIATED NONCOMMUTATIVE VECTOR BUNDLES

Piotr M. Hajac
(IMPAN / University of New Brunswick)

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David Pask, Aidan Sims and Bartosz Zieliński

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Free actions of compact quantum groups

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a coaction of H on A (or an action of the compact quantum group (H, Δ) on A) if

1. $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
2. $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

Definition (D. A. Ellwood)

A coaction δ is called free iff

$$\boxed{\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H}.$$

Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations.

The Peter-Weyl subalgebra

of A is $\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes_{\mathrm{alg}} \mathcal{O}(H) \,\}$.

The Peter-Weyl-Galois Theorem

Theorem (P. F. Baum, K. De Commer, P.M.H.)

Let A be a unital C-algebra equipped with an action of a compact quantum group (H, Δ). The following conditions are equivalent:*

1. *The action is free.*
2. *The action satisfies the Peter-Weyl-Galois condition.*
3. *The action is strongly monoidal.*

Put $B = A^{\mathrm{co}H} := \{a \in A \mid \delta(a) = a \otimes 1\}$ (coaction-invariants).

The Peter-Weyl-Galois condition

is the bijectivity of the canonical map
$\mathcal{P}_H(A) \otimes_B \mathcal{P}_H(A) \ni x \otimes y \longmapsto (x \otimes 1)\delta(y) \in \mathcal{P}_H(A) \otimes_{\mathrm{alg}} \mathcal{O}(H)$.

Let V and W be $\mathcal{O}(H)$-comodules (representations of (H, Δ)).

The strong monoidality

is the bijectivity of the natural map
$(\mathcal{P}_H(A)\Box V) \otimes_B (\mathcal{P}_H(A)\Box W) \longrightarrow \mathcal{P}_H(A)\Box(V \otimes_{\mathrm{alg}} W)$.

Main result

Theorem

Let (H,Δ) be a compact quantum group, A and A' (H,Δ)-C-algebras, B and B' the corresponding fixed-point subalgebras, and $f : A \to A'$ an equivariant *-homomorphism. Then, if the action of (H,Δ) on A is free and V is a finite-dimensional representation of (H,Δ), the induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies*

$$\boxed{(f|_B)_*\big([\mathcal{P}_H(A)\Box V]\big) = [\mathcal{P}_H(A')\Box V]}\,.$$

Proof outline: Note first that, if $\ell : \mathcal{O}(H) \to \mathcal{P}_H(A) \otimes \mathcal{P}_H(A)$ is a strong connection on $\mathcal{P}_H(A)$, then the equivariance of f implies that $\ell' := (f \otimes f) \circ \ell$ is a strong connection on $\mathcal{P}_H(A')$. Now take advantage of Chern-Galois theory to show that applying f componentwise to an idempotent matrix over B representing $\mathcal{P}_H(A)\Box V$ through ℓ is an idempotent matrix over B' representing $\mathcal{P}_H(A')\Box V$ through ℓ'. $\Box$

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NONCOMMUTATIVE GEOMETRY THE NEXT GENERATION

Paul F. Baum, Alan Carey, Piotr M. Hajac, Tomasz Maszczyk

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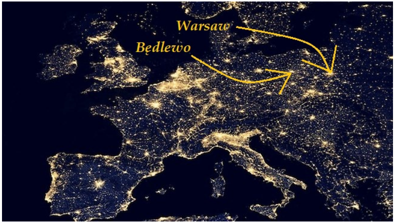

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$C(S^{2N+1})$ as a multi-pullback C*-algebra

Definition

The multi-pullback algebra A^π of a finite family $\{\pi^i_j : A_i \longrightarrow A_{ij} = A_{ji}\}_{i,j\in J,\, i\neq j}$ of algebra morphisms is defined as

$$A^\pi := \left\{ (a_i)_{i\in J} \in \prod_{i\in J} A_i \;\middle|\; \pi^i_j(a_i) = \pi^j_i(a_j),\ \forall\, i,j \in J,\, i \neq j \right\}.$$

$C(S^{2N+1})$ is isomorphic as a C*-algebra to the subalgebra of

$$\prod_{0\leq i\leq N} C(D)^{\otimes i} \otimes C(S^1) \otimes C(D)^{\otimes N-i}$$

defined by the compatibility conditions ($0 \leq i < j \leq N$, $\otimes$ suppressed):

$$C(D)^i C(S^1) C(D)^{N-i} \xrightarrow{\pi^i_j} C(D)^i C(S^1) C(D)^{j-i-1} C(S^1) C(D)^{N-j} \xleftarrow{\pi^j_i} C(D)^j C(S^1) C(D)^{N-j}.$$

Multi-pullback quantum spheres S_H^{2N+1}

$C(S_H^{2N+1})$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i}$ defined by the compatibility conditions prescribed by the following diagrams ($0 \leq i < j \leq N$, $\otimes$-supressed):

$$\mathcal{T}^i C(S^1) \mathcal{T}^{N-i} \xrightarrow{\sigma_j} \mathcal{T}^i C(S^1) \mathcal{T}^{j-i-1} C(S^1) \mathcal{T}^{N-j} \xleftarrow{\sigma_i} \mathcal{T}^j C(S^1) \mathcal{T}^{N-j}.$$

Here $\sigma_k := \mathrm{id}^k \otimes \sigma \otimes \mathrm{id}^{N-k}$ with domains and codomains determined by the context.

We equip all C*-algebras in the diagrams with the diagonal actions of $U(1)$. Since all morphisms in the diagrams are $U(1)$-equivariant, we obtain the diagonal $U(1)$-action on $C(S_H^{2N+1})$.

Gauging coactions

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a compact Hausdorff group G on a unital C*-algebra A. As with $U(1)$ acting on $\mathcal{T}$, we encode the G-action on A through the $C(G)$-coaction on A:

$$\rho : A \ni a \longmapsto a_{(0)} \otimes a_{(1)} \in A \otimes C(G) = C(G, A), \quad \rho(a)(g) := \alpha_g(a).$$

Here we use the Heyneman-Sweedler notation $\rho(a) =: a_{(0)} \otimes a_{(1)}$ for a convergent sum of simple tensors.

- $(A \otimes C(G))^D$ is the C*-algebra $A \otimes C(G)$ equipped with the diagonal coaction $a \otimes h \longmapsto a_{(0)} \otimes h_{(1)} \otimes a_{(1)} h_{(2)}$.
- $(A \otimes C(G))^R$ is the C*-algebra $A \otimes C(G)$ equipped with the coaction on the rightmost factor $a \otimes h \longmapsto a \otimes h_{(1)} \otimes h_{(2)}$.

G-equivariant C*-algebra isomorphisms:

$$F : (A \otimes C(G))^D \to (A \otimes C(G))^R, \quad a \otimes h \mapsto a_{(0)} \otimes a_{(1)} h,$$
$$F^{-1} : (A \otimes C(G))^R \to (A \otimes C(G))^D, \quad a \otimes h \mapsto a_{(0)} \otimes S(a_{(1)}) h,$$

where $S(h)(g) := h(g^{-1})$.

$C(S_H^{2N+1})$ as a gauged multi-pullback

The following diagrams ($0 \leq i < j \leq N$, $\otimes$ suppressed) are $U(1)$-equivariant with respect to the $U(1)$-actions on the rightmost factors.

$$
\begin{array}{ccccc}
i & \mathcal{T}^N C(S^1) & & \mathcal{T}^N C(S^1) & j \\
& \Big\downarrow{\scriptstyle \sigma_{j-1}} & & \Big\downarrow{\scriptstyle \sigma_i} & \\
& \mathcal{T}^{j-1}C(S^1)\mathcal{T}^{N-j}C(S^1) & \xleftarrow{\tilde{\Psi}_{ij}} & \mathcal{T}^{i}C(S^1)\mathcal{T}^{N-i-1}C(S^1), &
\end{array}
$$

$$
\tilde{\Psi}_{ij} : \bigotimes_{k=0}^{i-1} t_k \otimes v \otimes \bigotimes_{\substack{l=i+1 \\ l\neq j}}^{N} t_l \otimes w
$$

$$
\longmapsto \bigotimes_{\substack{k=0 \\ k\neq i}}^{j-1} t_{k(0)} \otimes S\left(\prod_{\substack{m=0 \\ m\neq i,j}}^{N} t_{m(1)}\right) S(v)w_{(1)} \otimes \bigotimes_{l=j+1}^{N} t_{l(0)} \otimes w_{(2)}.
$$

$C(S_H^{2N+1})$ is isomorphic as a $U(1)$-C*-algebra to the multi-pullback $U(1)$-C*-algebra of the above diagrams.

Quantum complex projective spaces $\mathbb{P}^N(\mathcal{T})$

$C(\mathbb{P}^N(\mathcal{T}))$ is the C*-subalgebra of $\prod_{i=0}^N \mathcal{T}^{\otimes N}$ defined by the compatibility conditions prescribed by the diagrams $(0 \le i < j \le N)$:

$$
\begin{array}{ccccc}
i & \mathcal{T}^{\otimes N} & & \mathcal{T}^{\otimes N} & j \\
& \Big\downarrow \sigma_j & & \Big\downarrow \sigma_{i+1} & \\
& \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j} & \xleftarrow{\Psi_{ij}} & \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i-1}, &
\end{array}
$$

$$
\Psi_{ij} : \bigotimes_{k=0}^{i-1} t_k \otimes v \otimes \bigotimes_{l=i+1}^{N-1} t_l \mapsto \bigotimes_{\substack{k=0\\k\neq i}}^{j-1} t_{k(0)} \otimes S\left(\Big(\prod_{\substack{m=0\\m\neq i}}^{N-1} t_{m(1)}\Big) v \right) \otimes \bigotimes_{l=j}^{N-1} t_{l(0)}.
$$

It follows from the gauged presentation of $C(S_H^{2N+1})$ that $C(\mathbb{P}^N(\mathcal{T})) \cong C(S_H^{2N+1})^{U(1)}$.

Universal presentation of $C(S_H^{2N+1})$

Let us define the following elements of $C(S_H^{2N+1})$:

$$a_i := \big((\sigma\otimes\mathrm{id}^{\otimes N})(1^{\otimes i}\otimes z\otimes 1^{\otimes N-i}), \ldots, (\mathrm{id}^{\otimes N}\otimes\sigma)(1^{\otimes i}\otimes z\otimes 1^{\otimes N-i})\big).$$

It straightforward to check that $\forall\, i,j \in \{0,\ldots,N\},\ i\neq j$:

$$a_i a_j = a_j a_i, \quad a_i a_j^* = a_j^* a_i, \quad a_i^* a_i = 1, \quad \prod_{i=0}^{N}(1 - a_i a_i^*) = 0.$$

Lemma (Key Lemma)

$C(S_H^{2N+1})$ is isomorphic as a $U(1)$-C-algebra with the universal C*-algebra generated by a_i's satisfying the above relations. The $U(1)$-action on the latter is given by rephasing the generators.*

Corollary

$C(S_H^{2N+1}) \cong \mathcal{T}^{\otimes N+1}/\mathcal{K}^{\otimes N+1}$, $K_0(C(S_H^{2N+1})) = \mathbb{Z}[C(S_H^{2N+1})] = \mathbb{Z}$, $K_1(C(S_H^{2N+1})) = \mathbb{Z}$.

Main application

Theorem

Let $L_k^{2N+1} := \{a \in C(S_H^{2N+1}) \mid \forall\, \lambda \in U(1) : \alpha_\lambda(a) = \lambda^k a\}$. Then $\forall\, N \in \mathbb{N} \setminus \{0\}$: $[L_m^{2N+1}] = [L_n^{2N+1}] \implies m = n$.

Proof outline:

1. By the preceding lemma, the assignments $a_k \mapsto b_k$ when $k < 2$ and $a_k \mapsto b_0$ when $k \geq 2$ define a $U(1)$-equivariant C*-homomorphism $f : C(S_H^{2N+1}) \to C(S_H^3)$. Here $a_0, \ldots, a_N$ are isometries generating $C(S_H^{2N+1})$ and b_0, b_1 are isometries generating $C(S_H^3)$.
2. Constructing a strong connection and applying the main theorem, we infer that the induced map
$$\big(f|_{C(\mathbb{P}^N(\mathcal{T}))}\big)_* : K_0\big(C(\mathbb{P}^N(\mathcal{T}))\big) \longrightarrow K_0\big(C(\mathbb{P}^1(\mathcal{T}))\big)$$
satisfies $\big(f|_{C(\mathbb{P}^N(\mathcal{T}))}\big)_*([L_m^{2N+1}]) = [L_m^3]$ for any $m \in \mathbb{Z}$.
3. Finally, as an index pairing computation proves that $[L_m^3] = [L_n^3] \Rightarrow m = n$ [P.M.H., R. Matthes, W. Szymański], the conclusion follows. □