Index, eta end rho invariants on stratified manifolds

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Primary invariants on closed orientable manifolds

- X orientable closed compact manifold; $\pi_1(X) := \Gamma$;
- X' Galois Γ-covering;
- $r : X \to B\Gamma$ a classifying map for X';
- we can consider the Fredholm index of the signature operator $\eth$ (if $\dim X = 2k$)
- more generally, the K-homology class $[\eth_{\mathrm{sign}}] \in K_*(X)$ of the signature operator;
- we can consider the signature operator with values in the Mischenko bundle $\mathcal{G}(r) := r^* E\Gamma \times_\Gamma C^*_r\Gamma$ and $\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{\mathrm{sign}}) \in K_*(C^*_r\Gamma)$,

Properties

- the index of the signature operator is the signature of the manifold (if $\dim X = 4k$)
- More generally, let $L_*(X) \in H_*(X, \mathbb{Q})$ be the homology L-class, i.e. the Poincaré dual of the Hirzebruch L-class in cohomology;
- then $\mathrm{Ch}_*[\eth_{\mathrm{sign}}] = L_*(X)$ (rationally);
- The index class $\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{\mathrm{sign}}) \in K_*(C^*_r \Gamma)$ is a homotopy invariant;

Eta invariants

The eta invariant associated to a Dirac operator D is by definition the value at $s = 0$ of the meromorphic continuation of

$$\eta(D)(s) := \frac{2}{\Gamma(\frac{(s+1)}{2})} \int_0^\infty t^{\frac{(s-1)}{2}} \mathrm{Tr}(D \exp(-tD^2) dt$$

- $\eta(D)$ is also the value at $s = 0$ of the meromorphic continuation of $\sum_{\lambda \neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s} \quad \mathrm{Re} s >> 0$.
- $\eta(D)$ measures the **spectral asymmetry** of the self-adjont op. D.
- $\eta(D)$ is a very sensitive invariant.
- Indeed, if $\{D_t\}$ is a one-parametr family of operators then (assuming for simplicity D_0 and D_1 invertible)

$$\eta(D_1) - \eta(D_0) = \int_M \mathrm{local} + \mathrm{SF}(\{D_t\})$$

Atiyah-Patodi-Singer rho invariant

- it is associated to the choice of a pair of finite dimensional unitary representations of $\pi_1(M) := \Gamma$ of the same dimension: $\lambda_1, \lambda_2 \colon \Gamma \to U(\mathbb{C}^N)$.
- we consider $L_j := \widetilde{M} \times_{\lambda_j} \mathbb{C}^N$
- we can *twist* D by L_j obtaining two operators D_{L_1} and D_{L_2}.
- then the Atiyah-Patodi-Singer rho invariant is by definition

$$\rho(D)_{\lambda_1 - \lambda_2} := \eta(D_{L_1}) - \eta(D_{L_2})$$

- this is a **more stable** invariant than eta itself
- particularly useful when $\pi_1(M)$ is a torsion group
- for example: in distinguishing metrics of Positive Scalar Curvature (PSC)
- for example: in distinguishing the diffeomorphism type of homotopically equivalent manifolds
- the rho-invariant is a **secondary invariant** (e.g.: the index for a positive scalar curvature metric is zero but rho is not)

The Cheeger-Gromov eta invariant

- $\widetilde{M} \to M$ a Γ-covering;
  we lift D to a Γ-invariant Dirac operator $\widetilde{D}$ on $\widetilde{M}$.
- The L^2-eta invariant is :

$$\eta_{(2)}(\widetilde{D}) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_{(2)}(\widetilde{D} \exp(-(t\widetilde{D})^2) dt$$

  where $\mathrm{Tr}_{(2)}$ is the Von Neumann trace introduced by Atiyah,(i.e.

$$\mathrm{Tr}_{(2)}(\widetilde{D} \exp(-(t\widetilde{D})^2) = \int_{\mathcal{F}} \mathrm{tr}_x \widetilde{K}_t(x,x). \qquad (1)$$

  with $\mathcal{F}$ a fundamental domain for $\widetilde{M} \to M$.
- of course should introduce the Von Neumann algebra of Γ-invariant bounded operators on $L^2(\widetilde{M}, \widetilde{E})$; it comes with a trace; these operators are trace class and their trace is given by (**??**)

Rho invariants

. The **Cheeger-Gromov** ρ-invariant is defined as the difference

$$\rho_{(2)}(\widetilde{D}) := \eta_{(2)}(\widetilde{D}) - \eta(D)$$

This is a **stable** invariant:

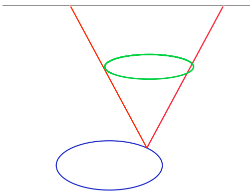
Example

If $\{g_t\}$ is a path of metrics of **P**ositive **S**calar **C**urvature on a spin manifold, then $\rho(\widetilde{D}_t^{\mathrm{spin}})$ is constant;

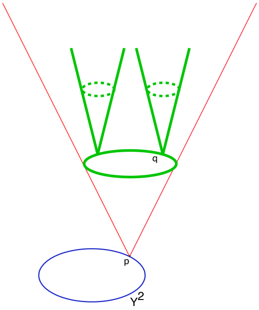
Example

On an orientable manifold $\rho(\widetilde{\eth}_{\mathrm{sign}})$ is independent of the metric and an oriented-diffeomorphism invariant.

Stratified pseudomanifolds



Above is an example of depth 1;below is an example of depth 2:



Very general questions

Let $\widehat{X}$ be a stratified pseudomanifold:

1. Is the signature operator Fredholm ?
2. Is there a signature K-homology class in $K_*(\widehat{X})$?
3. Is there a signature index class in $K_*(C_r^*\Gamma)$, $\Gamma = \pi_1(\widehat{X})$?
4. what are their properties ?
5. can one formulate and prove a Novikov Conjecture ?
6. are there secondary rho-type invariants ?

Witt spaces

We now restrict the class of pseudomanifolds. We consider Witt spaces:

Definition

$\widehat{X}$ is a Witt space if any even-dimensional link L has $IH_{\overline{m}}^{\dim L/2}(L;\mathbb{Q}) = 0$.

If $\widehat{X}$ is Witt **then** :

- there is a well defined intersection homology signature
- one can now define a homology L-class $L_*(\widehat{X}) \in H_*(\widehat{X};\mathbb{Q})$ à la Thom
- $\eth_{\rm sign}$ is essentially self-adjoint and Fredholm and the index is equal to the signature
- there is a well defined K-homology class $[\eth_{\rm sign}] \in K_*(\widehat{X})$ and $\mathrm{Ch}_*([\eth_{\rm sign}]) = L_*(\widehat{X})$
- the index class $\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{\mathrm{sign}}) \in K_*(C^*_r\Gamma)$ is well defined and it is a stratified homotopy invariant.

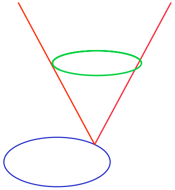
Cheeger spaces

We want to drop the Witt assumption and treat more general stratified spaces

$$\{\text{Witt spaces}\} \subset \{\text{Cheeger spaces}\} \subset \{\text{Stratified spaces}\}$$

Iterated conic metrics.

Let us concentrate on the depth one case.



So there is a decomposition of $\widehat{X}$ into two **strata**:
Y, the singular set (the bottom blue circle) and X, the regular part (this is the union of the red cones (without the vertices)).
The link of a point $p \in Y$ is a smooth closed manifold Z (the green circle).
A neighborhood of p looks like $B \times C(Z)$, with B a ball in $\mathbb{R}^{\dim Y}$.In fact a tubular neighborhood T of Y is a bundle of cones $C(Z) \to T \xrightarrow{\pi} Y$, as in the figure. A iterated conic metric on X is, by definition, the incomplete metric $g := dx^2 + x^2 g_Z + \pi^* h_Y$.
(Conic) analysis on $\widehat{X}$ means analysis on (X, g).

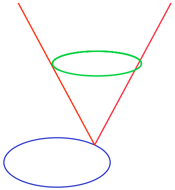
Preliminary problems

- we want to use Hilbert-space techniques
- we want closed operators
- if $\widehat{X}$ is Witt, then $d_{\min} = d_{\max}$ and $\eth_{\mathsf{sign}} : \Omega_c^+ \oplus \Omega_c^- \to \Omega_c^+ \oplus \Omega_c^- \to$ is essentially self-adjoint
- in the non-Witt case $d : \Omega_c^k \to \Omega_c^{k+1}$ has various closed extensions (between $d_{\min}$ and $d_{\max}$)
- similarly $\eth_{\mathsf{sign}}$ is NOT essentially self-adjoint

Resolution

- we resolve the pseudomanifold $\widehat{X}$ to a manifold with corners $\widetilde{X}$ (Verona + Brasselet-Hector-Saralegi + ALMP). $\widetilde{X}$ has an additional structure: it has an an iterated fibration structure on the boundary (boundary hypersurfaces are fibrations + compatibility relations at the corners between these fibrations).

Example: if $\widehat{X}$ is a depth-one space



then $\widetilde{X}$ is a manifold with boundary and the boundary is a fibration with base equal to the singular stratum (the bottom circle) and fiber the links (the green circles).

Expansions

We consider again $\eth_{\mathrm{dR}}$. Recall that a tubular neighborhood T of the singular set Y looks like

$$C(Z) \to T \to Y$$

Consider the resolved manifold $\widetilde{X}$; a manifold with boundary with boundary equal to the fibration $Z \to H \to Y$.
If Z is even-dimensional and has cohomology in middle degree then we are NOT in the Witt case.

Fundamental Lemma *Any $u \in \mathcal{D}_{\max}(\eth_{\mathrm{dR}})$ has an asymptotic expansion at Y,*

$$u \sim x^{1/2}(\alpha_1(u) + dx \wedge \beta_1(u)) + \widetilde{u}$$

with the terms in this expansion distributional:

$$\alpha_1(u), \beta_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes \mathcal{H}^{f/2}(H/Y)), \quad \widetilde{u} \in xH^{-1}(X, \Lambda^* X)$$

Here $\mathcal{H}^{f/2}(H/Y)$ is the flat Hodge bundle over Y (with typical fiber $\mathcal{H}^{f/2}(Z)$) and $f = \dim Z$

Cheeger boundary condition

The distributional differential forms $\alpha(u)$, $\beta(u)$ serve as 'Cauchy data' at Y which we use to define *Cheeger ideal boundary conditions*. Here is what we do: for any subbundle

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{H}^{f/2}(H/Y) \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

that is parallel with respect to the flat connection, we define

$$\mathcal{D}_W(\eth_{\mathrm{dR}}) = \{u \in \mathcal{D}_{\max}(\eth_{\mathrm{dR}}) :$$
$$\alpha_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes W), \quad \beta_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes (W)^{\perp})\}.$$

We call W a (Hodge) mezzoperversity adapted to g.

Inductive procedure

Let us go back to Cheeger boundary conditions. A collection of bundles

$$\mathcal{W} = \{W^1 \to Y^1, W^2 \to Y^2, \ldots, W^T \to Y^T\}$$

on the singular strata of $\widehat{X}$ is a **(Hodge) mezzoperversity** adapted to g if, inductively, each W^i is a flat subbundle of the Hodge bundle with the "previous" boundary conditions.

Techniques

With respect to a conic metric $dx^2 + x^2 g_Z + \pi^* h_Y$ we can write the de Rham operator $\eth_{\mathrm{dR}} = d + d^*$ near a singular stratum as (here $f = \dim Z$ and $\mathbf{N}$ is the number operator)

$$\eth_{\mathrm{dR}} \sim \begin{pmatrix} \frac{1}{x}\eth^Z_{\mathrm{dR}} + \eth^Y_{\mathrm{dR}} & -\partial_x - \frac{1}{x}(f - \mathbf{N}) \\ \partial_x + \frac{1}{x}\mathbf{N} & -\frac{1}{x}\eth^Z_{\mathrm{dR}} - \eth^Y_{\mathrm{dR}} \end{pmatrix}$$

with $\eth^Y_{\mathrm{dR}}$ and $\eth^Z_{\mathrm{dR}}$ the de Rham operators for h_Y and g_Z, respectively.

Main idea: consider $x\eth_{\mathrm{dR}}$ instead !
It can be considered on $\widetilde{X}$.
This is an elliptic differential operator in the edge-calculus of Mazzeo.
This brings us to microlocal techniques à la Melrose; the key-lemma and the analityc results that follow are all consequences of these techniques.

RESULTS. Part 1.

- Every mezzoperversity induces a closed self-adjoint domain $\mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{dR}})$;
- $(\eth_{\mathrm{dR}}, \mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{dR}}))$ is Fredholm with discrete spectrum;
- the are de Rham cohomology groups, $H^*_{\mathcal{W}}(\widehat{X})$, that are finite dimensional and metric independent;
- there is a Hodge decomposition theorem.

RESULTS. Part 2

- given a mezzoperversity $\mathcal{W}$ there is a dual mezzoperversity $\mathcal{D}\mathcal{W}$ defined in terms of the vertical Hodge-$\star$
- there is a natural non-degenerate pairing $H^{\ell}_{\mathcal{W}}(\widehat{X}) \times H^{n-\ell}_{\mathcal{D}\mathcal{W}}(\widehat{X}) \to \mathbb{R}$
- if $\mathcal{W} = \mathcal{D}\mathcal{W}$ then we say that $\mathcal{W}$ is self-dual (it might not exists)
- a space admitting a self-dual mezzoperversity is a **Cheeger space**;
- on a Cheeger space we have a non-degenerate pairing $H^{\ell}_{\mathcal{W}}(\widehat{X}) \times H^{n-\ell}_{\mathcal{W}}(\widehat{X}) \to \mathbb{R}$ and thus a signature $\sigma_{\mathcal{W}}(\widehat{X})$;
- a self-dual mezzoperversity defines a Fredholm signature operator $(\eth_{\rm sign}, \mathcal{D}_{\mathcal{W}}(\eth_{\rm sign}))$;
- the index is equal to the signature :

$$\sigma_{\mathcal{W}}(\widehat{X}) = \mathrm{ind}(\eth_{\rm sign}, \mathcal{D}_{\mathcal{W}}(\eth_{\rm sign})) \equiv \mathrm{ind}(\eth_{{\rm sign},\mathcal{W}})$$

- there is a well defined K-homology class $[\eth_{{\rm sign},\mathcal{W}}]$ in $K_*(\widehat{X})$
- if $\pi_1(\widehat{X}) = \Gamma$ and $r : \widehat{X} \to B\Gamma$ is a classifying map then we also have a higher index class

$$\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{{\rm sign},\mathcal{W}}) \in K_*(C^*_r\Gamma)$$

Summary+ 2 Crucial Questions

Given a Cheeger space $\widehat{X}$ with a fixed self-dual mezzoperversity $\mathcal{W}$ and a classifying map r we have defined

- $H^*_{\mathcal{W}}(\widehat{X})$
- $\sigma_{\mathcal{W}}(\widehat{X}) \in \mathbb{Z}$
- $[\eth_{\mathrm{sign},\mathcal{W}}]$ in $K_*(\widehat{X})$
- $\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{\mathrm{sign},\mathcal{W}}) \in K_*(C^*_r\Gamma)$

Question 1: what happens to these invariants if $F : \widehat{X} \to \widehat{M}$ is a stratified homotopy equivalence ??

Question 2: how does all this depend on the choice of $\mathcal{W}$??

Independence on $\mathcal{W}$

Theorem

$\sigma_{\mathcal{W}}(\widehat{M})$ and $\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{\mathrm{sign},\mathcal{W}})$ are independent of $\mathcal{W}$!

(This is a highly non-trivial result)!
Consequently: for a Cheeger space $\widehat{M}$ there is a homology L-class $L_*(\widehat{M}) \in H_*(\widehat{M}, \mathbb{Q})$.

Theorem

One can prove that $\mathrm{Ch}_*[\eth_{\mathrm{sign},\mathcal{W}}] = L_*(\widehat{M})$.
(Thus $[\eth_{\mathrm{sign},\mathcal{W}}]$ does not depend on $\mathcal{W}$ rationally.)

Theorem

The index class $\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{\mathrm{sign},\mathcal{W}}$ is not only independent of $\mathcal{W}$ but also a stratified homotopy invariant.

Primary invariants: summary

For index-theoretic invariants of the signature operator on Witt spaces and on Cheeger spaces we have seen that we have most of the classic properties:

- there is a well defined numeric index;
- there is a well defined index class which is a stratified homotopy invariant

For general Dirac-type operators the situation is much more complicated. For example for the spin-Dirac operator we have to require invertibility of the Dirac family along the link-fibration (recent paper of Albin and Jesse-Redman).

Secondary invariants

Here the theory is really at his beginning. Recent result :

Theorem

Let $\widehat{X}$ be a Witt space of odd dimension and of depth 1. Then the eta invariant of the signature operator is well defined. Consequently, the Atiyah-Patodi-Singer rho-invariant is also well defined.

This is joint work with Boris Vertman. We expect a similar result for the Cheeger-Gromov rho-invariant.

T H A N K Y O U